Fringe Field of Solenoid*

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*This subject was suggested to study by E. Perevedentsev.

1 Model of Magnetic Field

We consider a longitudinal solenoid field with an axial symmetry:

$$B_z(s) = \begin{cases} 0, & s < -f/2 \\ B_0(s/f + 1/2), & -f/2 \le s \le f/2 \\ B_0, & f/2 < s \end{cases}, \tag{1}$$

where f is the length of the fringe.

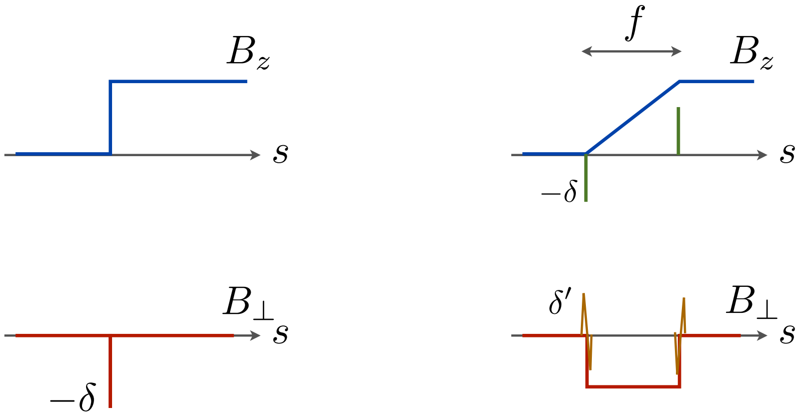

Figure 1: The magnetic field models: old (left) and new (right).

The associated vector potentials for $s \leq f/2$ are

$$
\begin{aligned}
A_x &= -\frac{B_0 y}{2}\max(s+f/2,0) + \frac{B_0(x^2+y^2)y}{16f}\left[\delta(s+f/2) - \delta(s-f/2)\right] \ , && (2)\\
A_y &= \frac{B_0 x}{2}\max(s+f/2,0) - \frac{B_0(x^2+y^2)x}{16f}\left[\delta(s+f/2) - \delta(s-f/2)\right] \ , && (3)\\
A_z &= 0 \ , && (4)
\end{aligned}
$$

where the terms with δ-functions are necessary in order to satisfy the Maxwell equations, while keeping the axial symmetry.

The magnetic field derived from the δ-function terms in (2) and (3) are

$$
\begin{aligned}
B_x &= -\frac{\partial A_y}{\partial s} = \frac{B_0(x^2+y^2)x}{16f}\left[\delta'(s+f/2) - \delta'(s-f/2)\right] \ , && (5)\\
B_y &= \frac{\partial A_x}{\partial s} = \frac{B_0(x^2+y^2)y}{16f}\left[\delta'(s+f/2) - \delta'(s-f/2)\right] \ , && (6)\\
B_z &= \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = -\frac{B_0(x^2+y^2)}{4f}\left[\delta(s+f/2) - \delta(s-f/2)\right] \ . && (7)
\end{aligned}
$$

Thus the Lorentz force gives the changes in momenta at $s = -f/2$ as:

$$
\begin{aligned}
p_x' &= \frac{p_y}{p}B_z - B_y = -\frac{p_y}{p}\frac{B_0(x^2+y^2)}{4f}\delta(s+f/2) - \frac{B_0(x^2+y^2)y}{16f}\delta'(s+f/2) && (8)\\
&= \frac{B_0}{16f}\left[-(3x^2+y^2)\frac{p_y}{p} + 2xy\frac{p_x}{p}\right]\delta(s+f/2) && (9)\\
p_y' &= -\frac{p_x}{p}B_z + B_x = \frac{p_x}{p}\frac{B_0(x^2+y^2)}{4f}\delta(s+f/2) + \frac{B_0(x^2+y^2)y}{16f}\delta'(s+f/2) && (10)\\
&= \frac{B_0}{16f}\left[(x^2+3y^2)\frac{p_x}{p} - 2xy\frac{p_y}{p}\right]\delta(s+f/2)\ , && (11)
\end{aligned}
$$

where we have used $x' \approx p_x/p$ and $y' \approx p_y/p$.

The equations of motion (9) and (11) are derived from a Hamiltonian:

$$D = \frac{B_0}{16fp}(x^2+y^2)(xp_y - yp_x)\left[\delta(s+f/2) - \delta(s-f/2)\right]\ . \tag{12}$$

2 Solution

The fringe field has at least two effects, linear and nonlinear. The linear effect is caused by the first terms in Eqs. (2) and (3) that are linear in x and y. We can expect that such linear effects can be expressed by a model with hard edges sliced along s, if the number of slices is sufficiently large. Thus here we concentrate on the nonlinear effects that are caused by the δ-function terms in Eqs. (2) and (3), or their Hamiltonian (12).

Let us obtain the transformation associated with the nonlinear terms up to the first order of B_0. It is expressed as

$$\exp(:-f/2:)\exp(:\delta:)\exp(:f:)\exp(:-\delta:)\exp(:-f/2:)\ , \tag{13}$$

where $\exp(:-f/2:)$ is a drift-back by a distance $-f/2$, and $\exp(:\delta:)$ is the nonlinear term at $s=-f/2$, etc.

Then the transformation (13) is approximated as

$$\begin{pmatrix} x_1 \\ p_{x1} \\ y_1 \\ p_{y1} \end{pmatrix} = \begin{pmatrix} x_0 + b\left(2p_{x0}x_0y_0 - p_{y0}(x_0^2-y_0^2)\right)/8p^2 \\ p_{x0} + b\left(2p_{x0}p_{y0}x_0 - (p_{x0}^2-p_{y0}^2)y_0\right)/8p^2 \\ y_0 - b\left(2p_{y0}x_0y_0 + p_{x0}(x_0^2-y_0^2)\right)/8p^2 \\ p_{y0} - b\left(2p_{x0}p_{y0}y_0 + (p_{x0}^2-p_{y0}^2)x_0\right)/8p^2 \end{pmatrix}\ , \tag{14}$$

where $b \equiv B_0/(B\rho)$, up to the first order of B_0.

The transformation (14) is expressed as $\exp(:H:)$ with a Hamiltonian:

$$H = -\frac{b}{8p^2}(xp_y - yp_x)(xp_x + yp_y)\ . \tag{15}$$

It is interesting that the transformation (14) and thus the Hamiltonian (15) are independent on the length of fringe, f.

To solve Eq. 15), it is convenient to use another set of variables:

$$\begin{pmatrix} r \\ p_r \\ \varphi \\ p_\varphi \end{pmatrix} = \begin{pmatrix} \log(x^2+y^2)/2 \\ xp_x + yp_y \\ \tan^{-1}(y/x) \\ xp_y - yp_x \end{pmatrix} , \tag{16}$$

which is generated by a generating function:

$$G(x, p_r, y, p_\varphi) = \frac{\log(x^2+y^2)}{2} p_r + \tan^{-1}\left(\frac{y}{x}\right) p_\varphi \ . \tag{17}$$

Then the Hamiltonian (15) is rewritten as

$$H = -\frac{b}{8p^2} p_\varphi p_r \ . \tag{18}$$

The transformation with (18) is simply written as:

$$\begin{pmatrix} r_1 \\ \varphi_1 \end{pmatrix} = \begin{pmatrix} r_0 - bp_\varphi/8p^2 \\ \varphi_0 - bp_r/8p^2 \end{pmatrix} , \tag{19}$$

where p_r and p_φ are constant.

The effect of the nonlinearity, given by Eq. (19), is more or less negligible for the present KEKB optics. It has significant impact, however, on the SuperKEKB lattices, by driving the betatron tunes into half-integer resonances. It is due to the larger orbit angle against the solenoid axis and the higher sensitivity with larger β at the solenoid fringe. Also it is reported that it affected the dynamic aperture and thus the Touschek lifetime was reduced by about 20% (Y. Ohnishi).

3 Synchrotron Radiation due to Solenoid Fringe

The transverse field $B_\perp$ at a fringe of solenoid causes synchrotron radiation. It is evaluated with the new model of solenoid field shown in Fig. 1. SAD was modified as following, but it should not be so special compared to usual radiation in dipole magnets:

Deviation from closed orbit ($\overline{p_x}$) is expressed as:

$$\delta p_x = p_x - \overline{p_x} \ . \tag{20}$$

Change of momenta due to radiation:

$$p' = p + \Delta p \ , \tag{21}$$

$$p'_x = p_x + \frac{p_x}{p}\Delta p \ , \tag{22}$$

$$\delta p'_x = \delta p_x + \frac{\delta p_x + \overline{p_x}}{p}\Delta p \ . \tag{23}$$

Change of beam matrix (quadratic moment) due to radiation:

$$\langle p'^2 \rangle = \langle p^2 \rangle + \langle \Delta p^2 \rangle \ , \tag{24}$$

$$\langle \delta p'^2_x \rangle = \left(1 + \frac{\langle \Delta p^2 \rangle}{p^2}\right) \langle \delta p_x^2 \rangle + \left(\frac{\overline{p_x}}{p}\right)^2 \langle \Delta p^2 \rangle \ , \tag{25}$$

$$\langle p' \delta p'_x \rangle = \left(1 + \frac{\langle \Delta p^2 \rangle}{p^2}\right) \langle p \delta p_x \rangle + \left(\frac{\overline{p_x}}{p}\right) \langle \Delta p^2 \rangle \ . \tag{26}$$

These are similar for y-plane.

The vertical emittance growth due to the fringe field of solenoid is non-negligible as shown in Table 1. The angle between the orbit and the solenoid axis should be chosen to minimize this effect, and actually the equal angle (41.5 mrad) for both rings is the optimum (Y. Ohnishi, A. Morita).

Table 1: Examples of emittance generated by solenoid fringe for SuperHER.

Optics	ε_x (nm)	ε_y (pm)
herfqlc4038.sad (67.5 mrad)	1.68	155.3
herfqlc4039-Sol-52.818mrad.sad	1.69	13.1
herfqlc4039-Sol-41.500mrad.sad	1.70	6.72

Problem was that the last terms in Eqs. (25) and (26) have been missed in SAD probably since around 1988. We noticed it as the closed orbit was large in a solenoid, where the reference coordinate was chosen on the axis of solenoid. Then we realized that this bug had been even more serious in the evaluation of emittance in wigglers in the LER, where the orbit excursion is also large, since the reference coordinate is chosen at the straight line through the wiggler section.

After the correction of the bug, we realized that *the present wiggler lattice was not usable for Super-LER*, since it increased the emittance to 5.8 nm[1], while the design required 2.6 nm (both numbers

[1]The corrected value of the emittance was also confirmed by E. Levichev and P. Piminov with an independent code.

were without intrabeam scattering). Then a new wiggler lattice was designed by H. Koiso, considering maximal reusal of the existing wiggler magnets.

The impact of the bug on the emittance for the present KEKB was not negligible but small: 18 nm with the bug vs. 19 nm after correction.