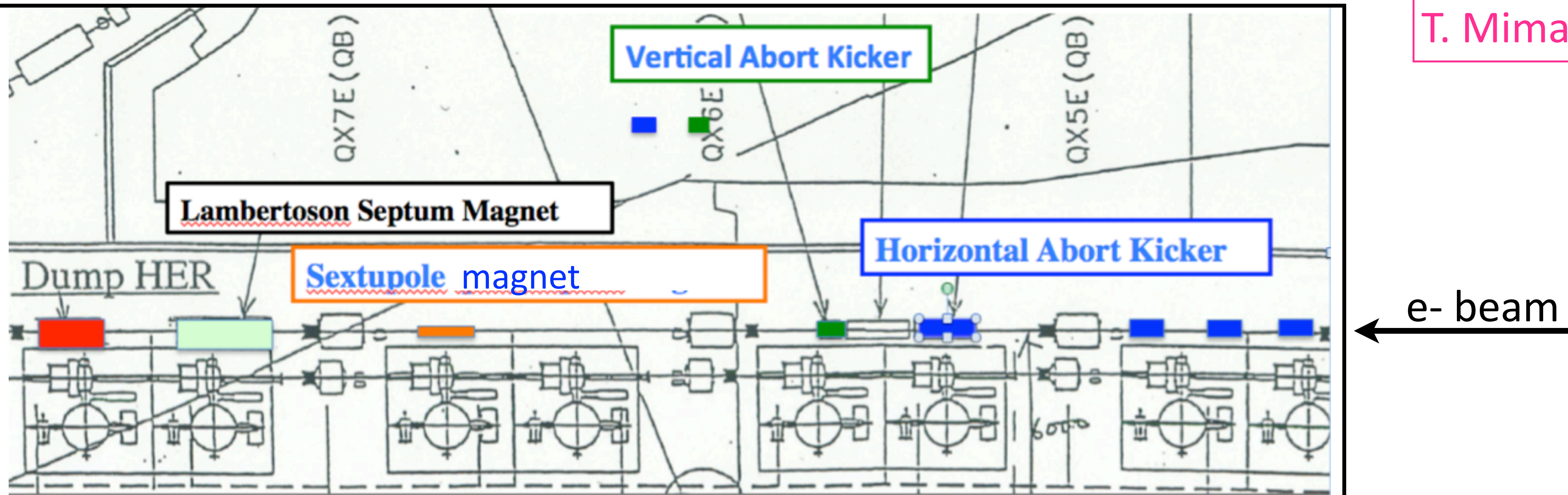
# Status of Abort System

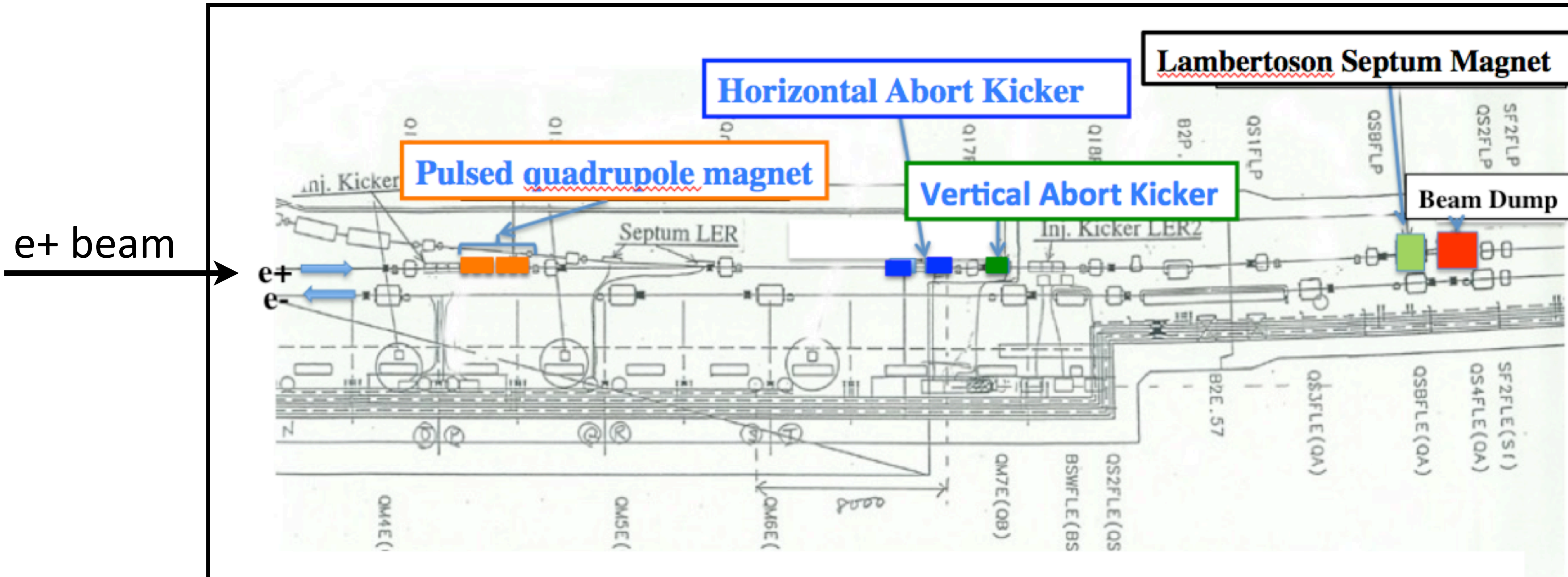
M. Kikuchi, KEKB Review, 24Feb2015

# Abort system, overview

T. Mimashi




HER Beam Abort System




LER Beam Abort System

# Abort system, Kickers

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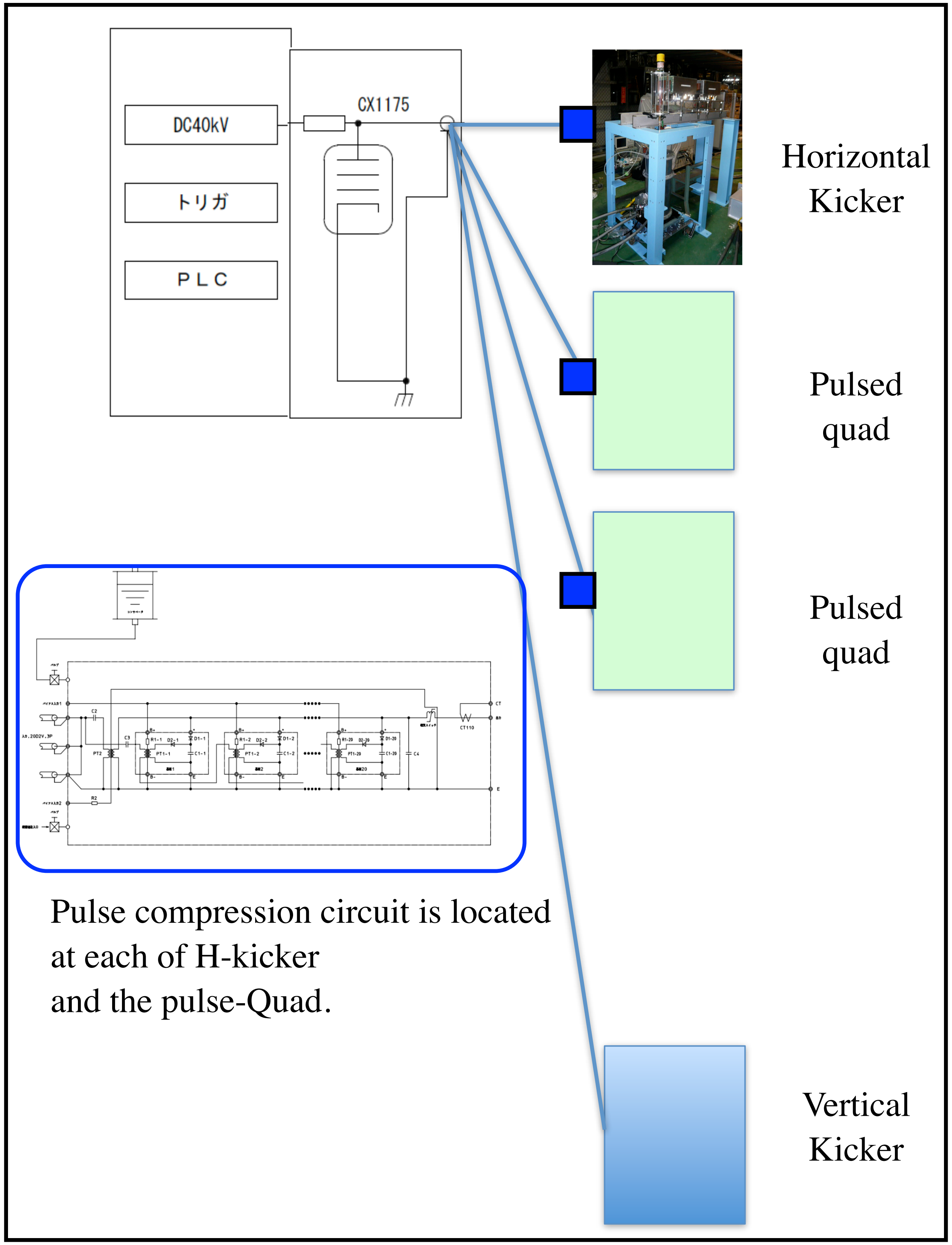
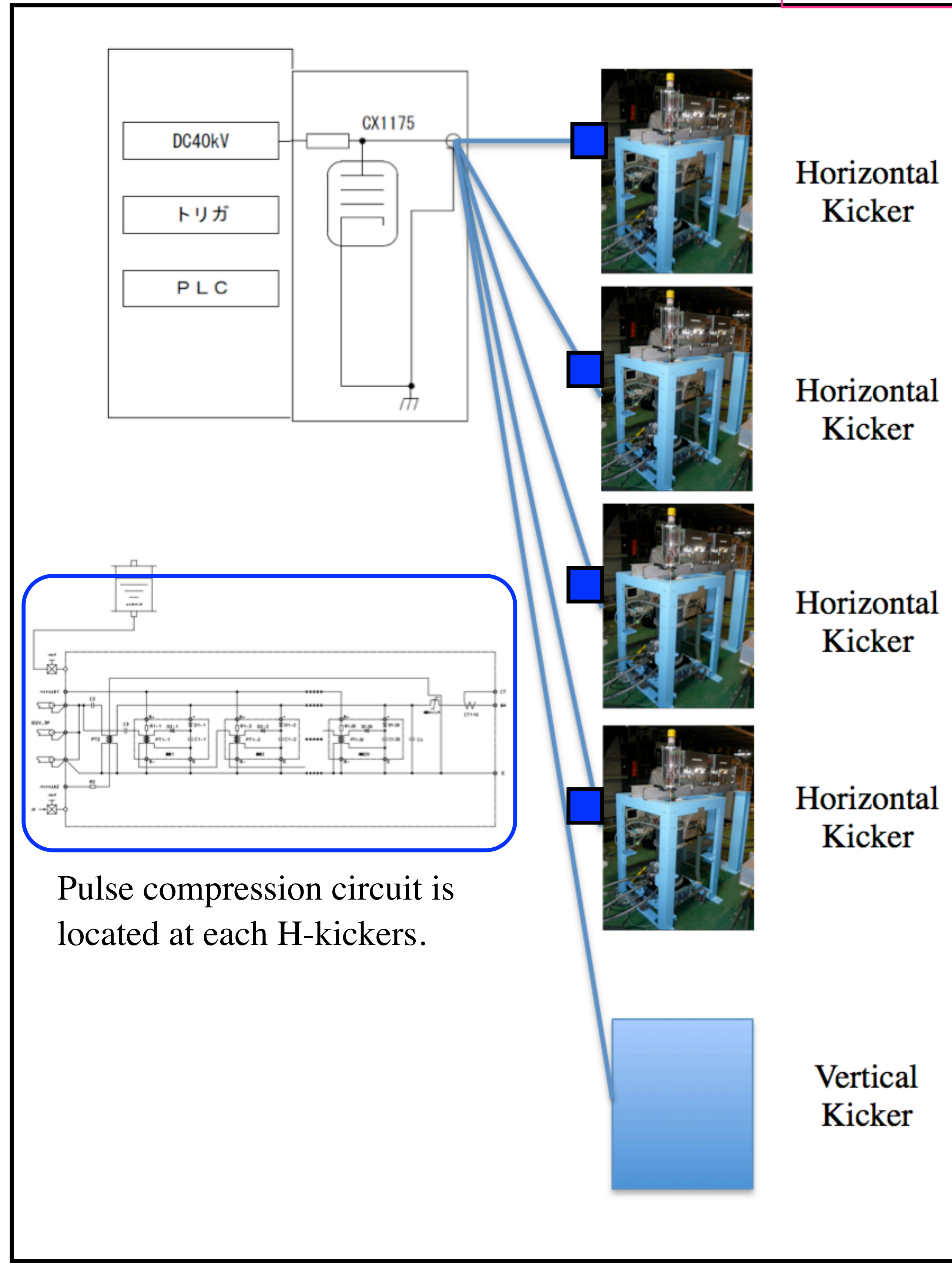



Pulse compression circuit is located at each of H-kicker and the pulse-Quad.

LER




Pulse compression circuit is located at each H-kickers.

HER

# Abort system: Waveform of the output current

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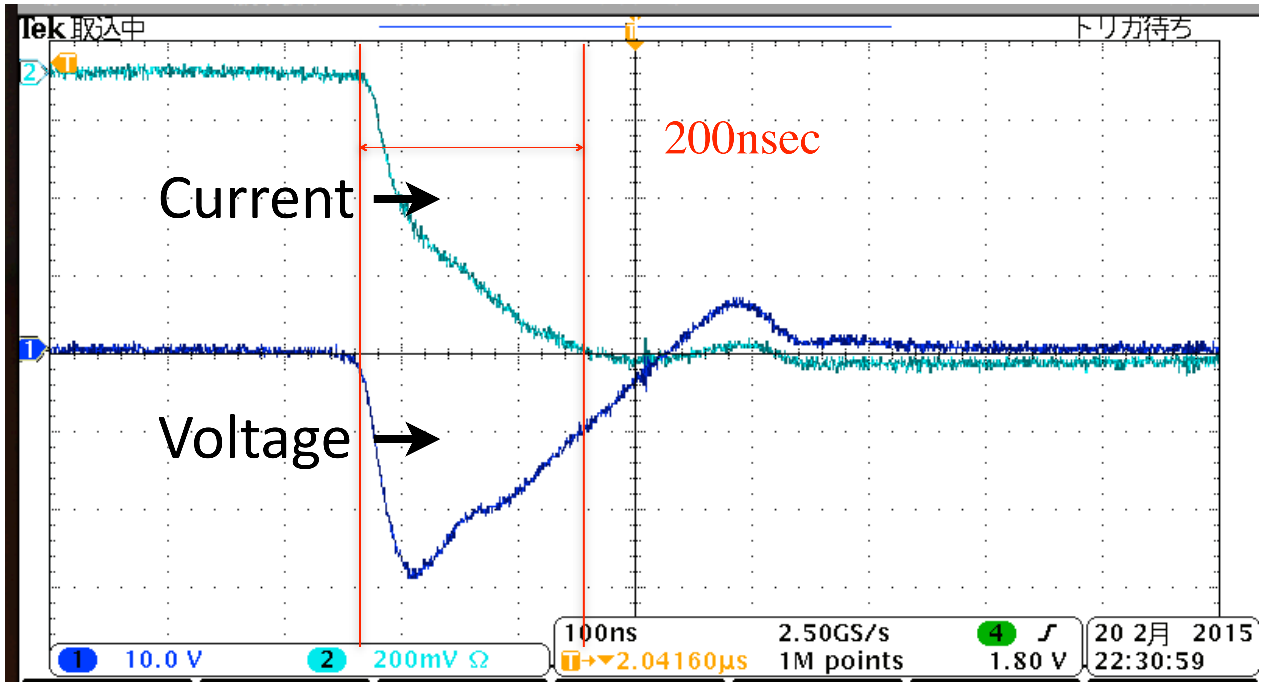
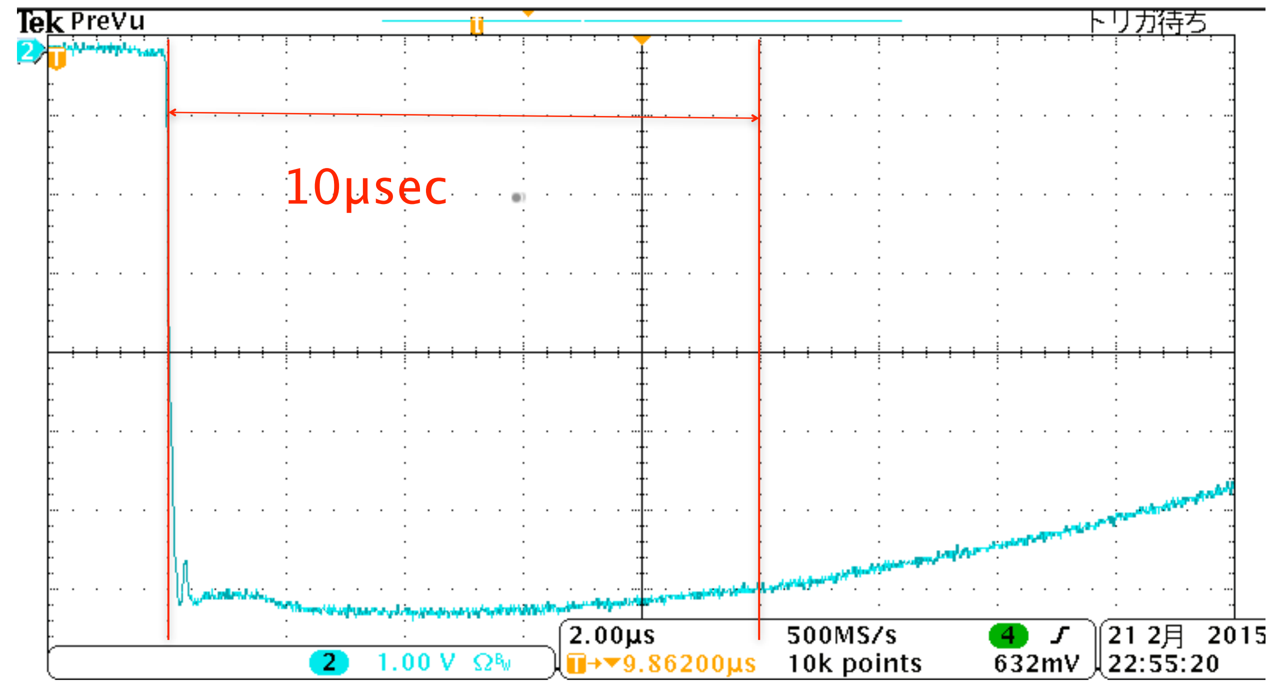
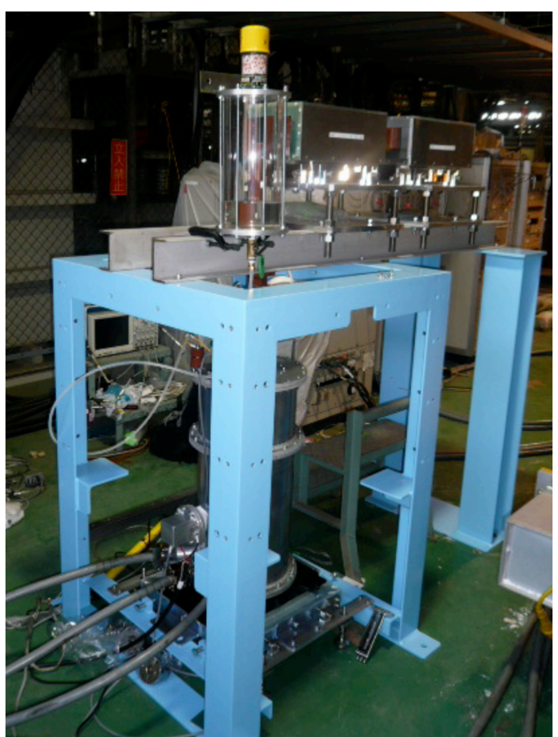
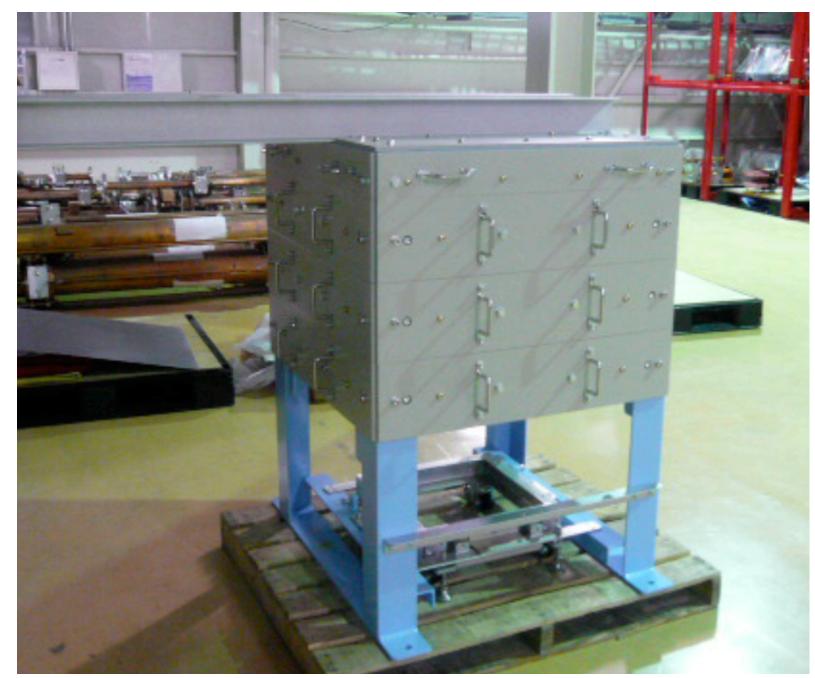



Current waveform of the horizontal kicker




Current waveform of the vertical kicker





Radiation shield of pulse compression circuit

# Abort system: installation

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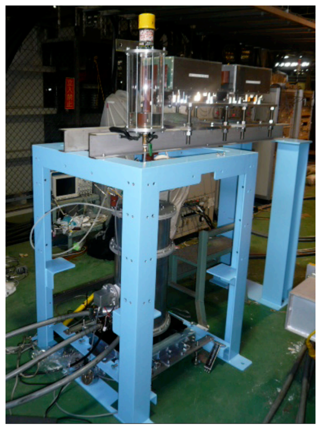
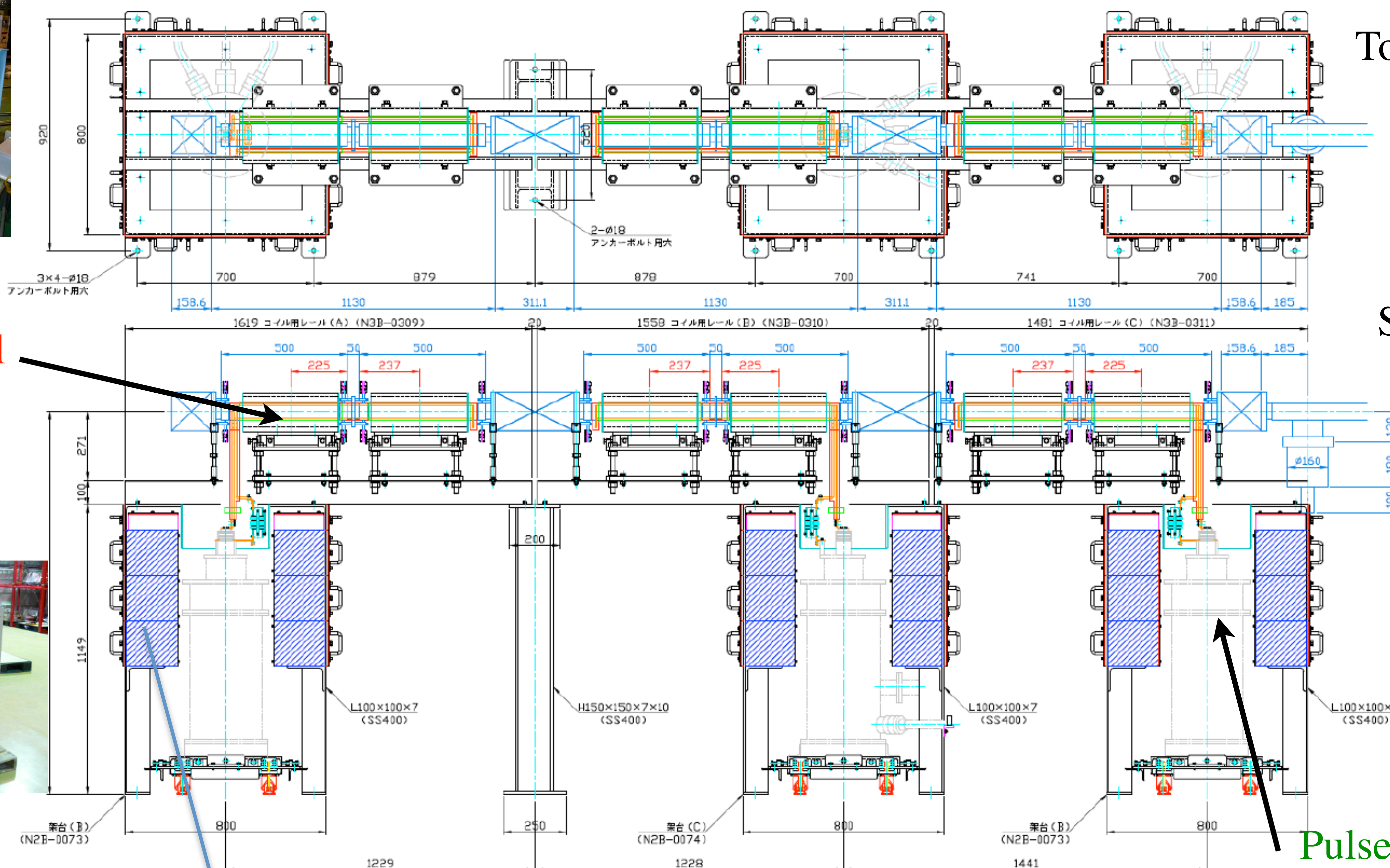
Layout of abort kicker magnets & pulse compression circuit at HER

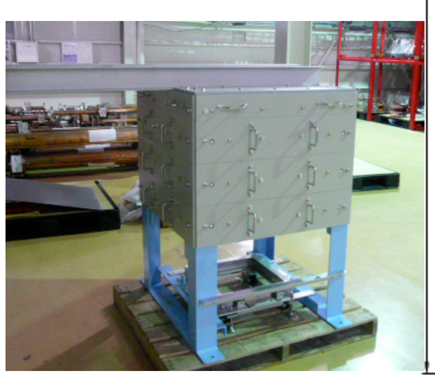


Top View

Side View

Horizontal Kicker Magnets



Pulse Compression circuit

Radiation shield: 7mm Pb + 200mm polyethylene + 1mm Pb

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•Radiation shield is necessary to protect sensitive semiconductors in the pulse compression circuit which is placed just below the beam line.

- In KEKB operation
  - Diodes for shaping the current waveform has survived, which have been placed just below the beam line.
- The radiation shield is designed to reduce the radiation to 1/10 at the surface of diodes (for both of $\gamma$ and neutrons).

# Abort system: Water-cooled ceramic chamber

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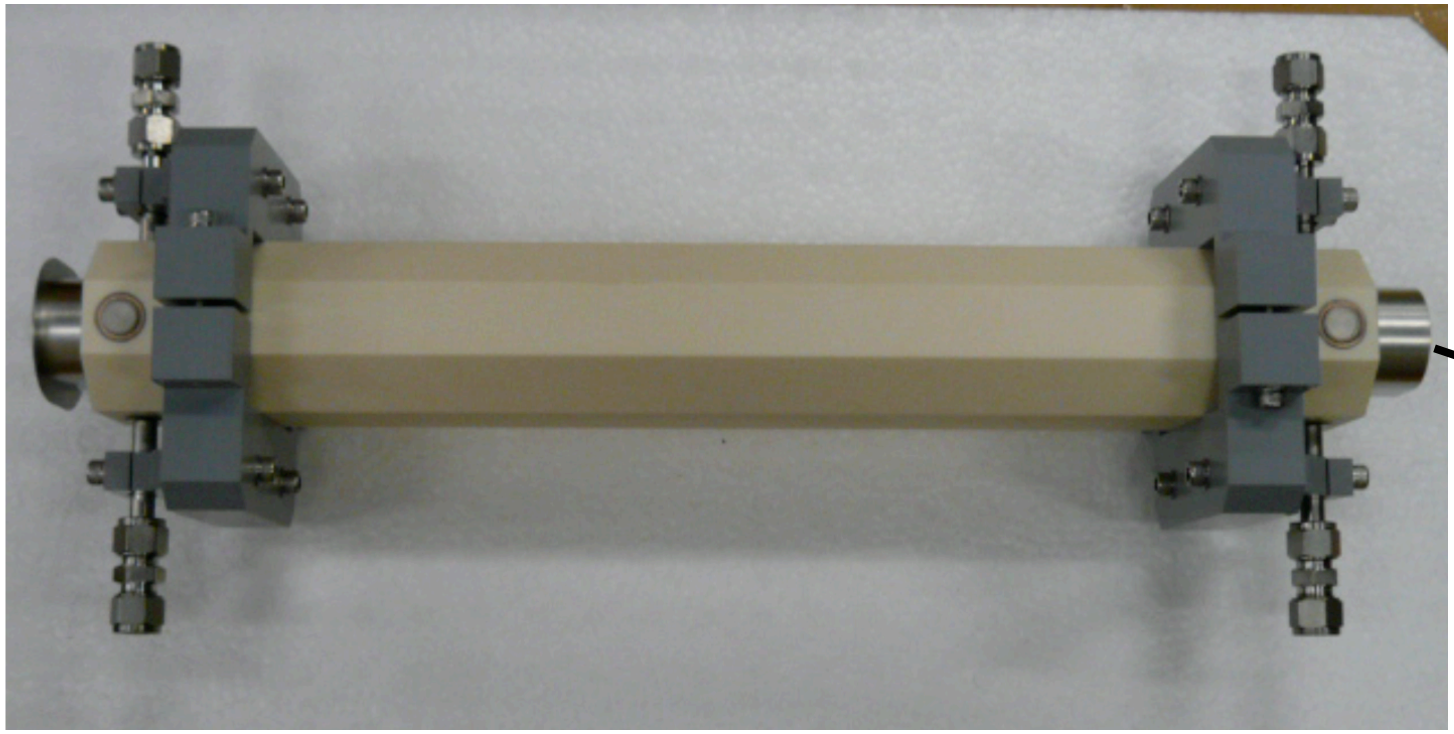
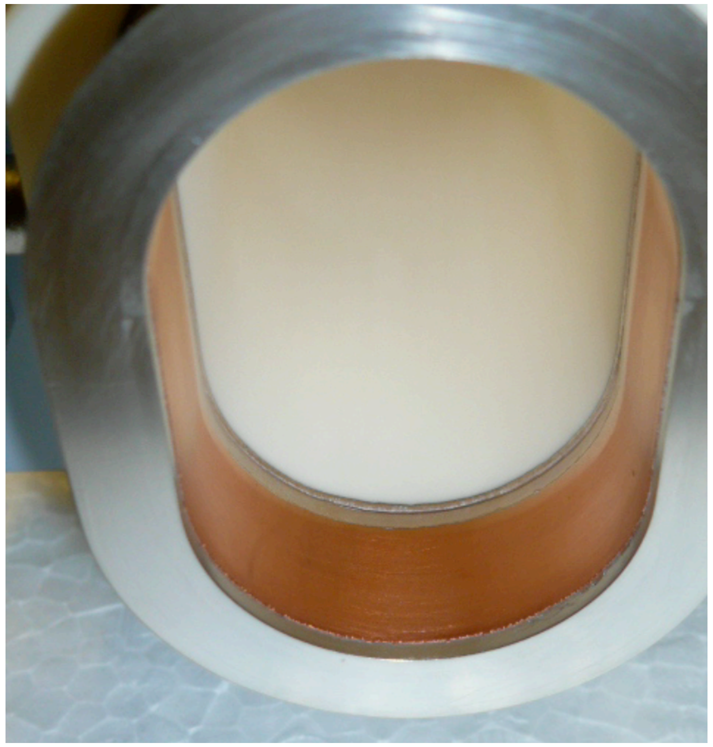
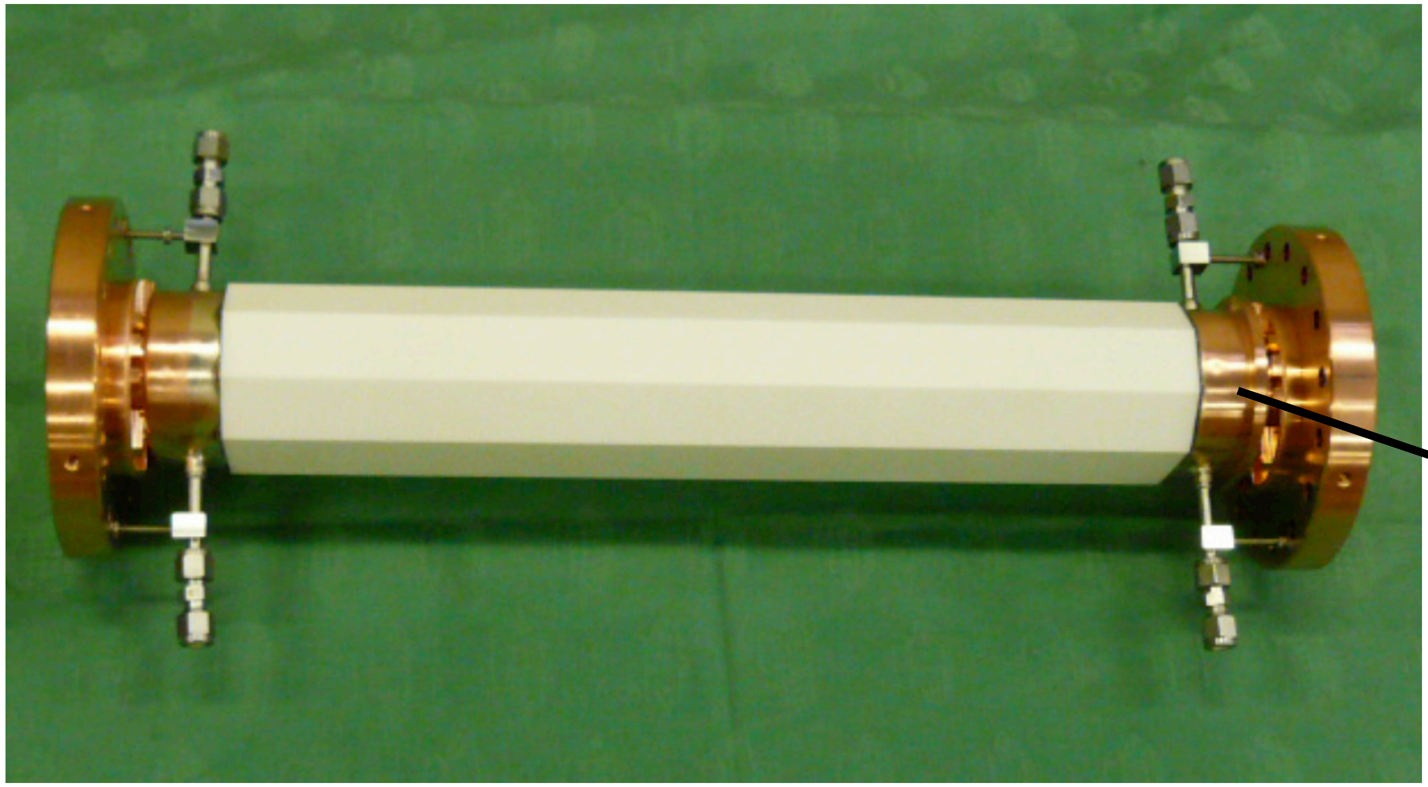


HER abort kicker ceramic chamber



100μm Cu-coating applied inside the Kovar sleeve



LER abort kicker ceramic chamber

Cu sleeve is directly connected to the ceramic

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# Fabrication status and plan

| | R&D, Design | | Fabrication | | Installation | |
|---|---|---|---|---|---|---|
| | HER | LER | HER | LER | HER | LER |
| H kickers | ○ | ○ | ○ | 2015 | 2015 | 2016 |
| V kickers | ○ | ○ | △(Mar, 2015) | Mar, 2015 | 2015 | 2016 |
| Pulsed quads | - | ○ | - | 2015 | - | 2016 |
| Ceramic Chambers | ○ | ○ | △(Ti coat) | ○ | 2015 | 2016 |
| Power supplies | ○ | ○ | △ | △ | 2015 | 2016 |
| Radiation shield | ○ | ○ | △ | 2015 | 2015 | 2016 |
| System Tuning | - | - | × | 2015 | - | - |

# Abort system: Summary

- Summary of the construction status of abort system
  - New power supplies for kickers were successfully developed.
    - Pulse compression using the magnetic switch
    - Satisfies the rise time of 200 ns.
    - Radiation shield was developed to protect the diodes in the pulse compressor.
  - New water-cooled ceramic chambers have been developed.
    - HER: cooling channels in the ceramic, Cu-coated Kovar sleeve.
    - LER: double concentric ceramic pipes connected both ends, Cu sleeve.
  - Fabrication is steadily going on, coping with tight schedule.

# Dynamic Stress in the Extraction Window of SuperKEKB

M. Kikuchi, KEKB Review, 24Feb2015

## Temperature rise

$$\Delta T = \frac{1}{C}\left(\frac{1}{\rho}\frac{dE}{dx}\right)\frac{N}{2\pi\sigma_x\sigma_y}$$

ionization loss for Ti

$$\frac{1}{\rho}\frac{dE}{dx} = 1.98 \text{ MeV}/(\text{g}/\text{cm}^2) \qquad C = 0.54 \text{ J}/\text{K}/\text{g}$$

$$\sigma_x \sim 0.3 \text{ mm} \qquad \sigma_y \sim 18\ \mu\text{m} \qquad eN = 36\ \mu C \text{ (LER)}$$

then $\Delta T = 4 \times 10^5 K$ !!

- We must reduce the power density
  - vertical beam-scan: 15 mm /10μs
  - enlarge σx by a pulsed-quad(LER), or sext w/offset(HER): $\sigma_x \longrightarrow 1.23 \text{ mm}$

$$\Delta T = \frac{1}{C}\left(\frac{1}{\rho}\frac{dE}{dx}\right)\frac{N}{\sqrt{2\pi}\sigma_x h}$$

$$\Delta T \sim 300 \text{ K}$$

# Dynamic Stress in the Extraction Window

- Related parameters

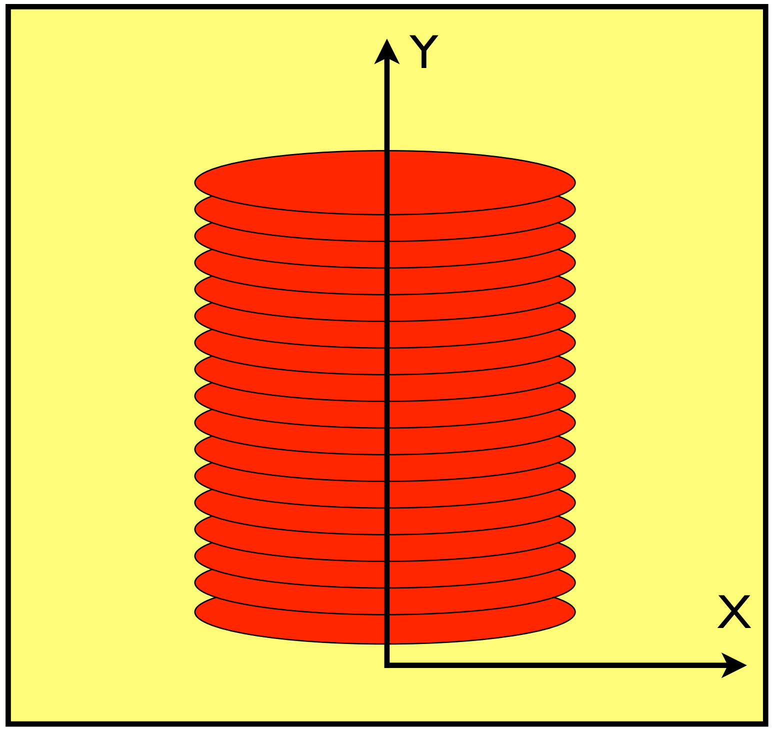
  - Beam parameters

    $$\sigma_x = 1.23\,\mathrm{mm}, \quad \sigma_\mathrm{y} = 18\,\mu\mathrm{m}$$

  - Bunch space : $\Delta t = 4\,\mathrm{ns}$

  - Scan speed :

    $$\Delta y/\Delta t = 15\,\mathrm{mm}/(2500\Delta\mathrm{t}) = 6\,\mu\mathrm{m}/4\,\mathrm{ns}$$




  - Window material : Ti

    $$E = 106\,\mathrm{GPa}, \quad \nu = 0.34, \quad \rho = 4.5\,\mathrm{g/cm}^3 \qquad \alpha = 8.4\times 10^{-6}$$

  - Speed of sound $c = \sqrt{E/\rho/(1-\nu^2)} = 5160\,\mathrm{m/s}$

  - Propagation length / bunch : $c\Delta t = 21\,\mu\mathrm{m}$

# Dynamic Stress in the Extraction Window

- Maximum temperature rise for a single bunch

$$T_0 = \frac{1}{C}\left(\frac{1}{\rho}\frac{dE}{dx}\right)\frac{N}{2\pi\sigma_x\sigma_y}$$

$$\boxed{T_0 = 39 \text{ K/bunch}}$$

$$eN = 14.8 \text{ nC/bunch}$$

- Thermal Stress

$$\boxed{\sigma_0 \equiv \frac{\alpha T_0 E}{1-\nu} = 52.6 \text{ MPa}}$$

- Temperature due to multi-bunch

$$T(y) = T_0 \sum_{n=0}^{M-1} \exp\left(-\frac{(y-n\Delta y)^2}{2\sigma_y^2}\right) \simeq T_0 \frac{\sqrt{2\pi}\sigma_y}{\Delta y} \mathrm{erf}(\mathrm{y}/\sigma_{\mathrm{y}})$$

- Temperature saturates at 293 K in several bunches

$$\boxed{T_{max} = T_0 \frac{\sqrt{2\pi}\sigma_y}{\Delta y} = 7.5 T_0 = 293 \text{ K}}$$

# Dynamic Stress in the Extraction Window

- Maximum thermal stress due to multi-bunch

$$\sigma_{max} = \frac{T_{max}}{T_0}\sigma_0 = 7.5\sigma_0 = 395 \text{ MPa ?}$$

❋ Obviously  this is incorrect because,

- Stress wave moves much faster than the scanned beam

Basic question:

- What is the dynamic stress for a Gaussian beam?
- How does it behave in the flat Gaussian beam?

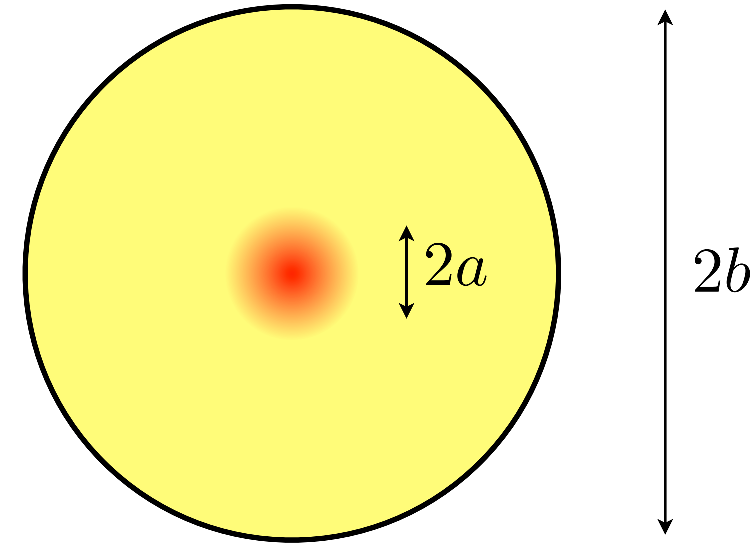
•Summary of related parameters

| | | SLER | LER | |
|---|---|---|---|---|
| Beam current | $I$ | 3.6 | 1.64 | A |
| Number of bunches | $n_b$ | 2500 | 1584 | |
| Charge per bunch | $eN$ | 14.8 | 10.4 | nC/bunch |
| Hor. emittance | $\varepsilon_x$ | 3.2 | 18 | nm |
| Ver. emittance | $\varepsilon_y$ | 8.6 | 150 | pm |
| Hor. beam-size at window | $\sigma_x$ | 1.23 | 0.70 | mm |
| Ver. beam-size at window | $\sigma_y$ | 18 | 73 | μm |
| Bunch spacing | $\Delta t$ | 4 | 6 | ns |
| propagation/bunch | $c\Delta t$ | 21 | 31.5 | μm |
| Scan height | $h$ | 15 | 15 | mm |
| scan height / bunch | $\Delta y$ | 6 | 9 | μm |
| Temp. rise per bunch | $T_0$ | 39 | 12 | deg |
| Maximum temp. rise | $T_{max}$ | 293 | 230 | deg |

# Model I.

## Model I.

- Thin disk
- Cylindrical symmetry
- Single Gaussian beam




- Instantaneous stress induced by abrupt change of temperature is propagated as a 'stress-wave' with a speed of sound

Position deviation U and stresses $\sigma_r, \sigma_\theta$ must satisfy the Eq. of motion

$$\rho \frac{d^2U}{dt^2} = \frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} \qquad (1)$$

U(r,t) : (radial) position deviation

From Hooke's law

$$\varepsilon_r - \alpha T = \frac{1}{E}(\sigma_r - \nu \sigma_\theta) \qquad (2)$$

T(r) : temperature

$$\varepsilon_\theta - \alpha T = \frac{1}{E}(\sigma_\theta - \nu \sigma_r) \qquad (3)$$

and definitions of strains

$$\varepsilon_r = \frac{\partial U}{\partial r}, \quad \varepsilon_\theta = \frac{U}{r} \qquad (4)$$

Ref.
P. Sievers, *Elastic stress waves in matter due to rapid heating by an intense high-energy particle beam*, LAB. II/BT/74-2, 26 June 1974

## Model I. Eq. of Motion

The Eq. of motion is rewritten as

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} - \frac{U}{r^2} - (1+\nu)\alpha\frac{dT}{dr} = \frac{1}{c^2}\frac{\partial^2 U}{\partial t^2} \qquad (5)$$

$$c = \sqrt{\frac{E}{(1-\nu^2)\rho}}$$

: Speed of sound

Put $U = u + \bar{u}$ with a static solution $\bar{u}$ of

$$\frac{d^2\bar{u}}{dr^2} + \frac{1}{r}\frac{d\bar{u}}{dr} - \frac{\bar{u}}{r^2} - (1+\nu)\alpha\frac{dT}{dr} = 0 \qquad (6)$$

u (time-dependent part of U) satisfies the wave-eq.:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \qquad (7)$$

Solve (7) with initial conditions,

$$U(r,0) = 0, \quad i.e., \quad u(r,0) = -\bar{u}(r) \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0 \qquad (8)$$

and boundary conditions,

$$u(b,t) = \bar{u}(b,t) = 0. \qquad (9)$$

# Model I. Solution

Solution:

$$u(r,t) = (1+\nu)\alpha T_0 \sum_{n=0}^{\infty} c_n J_1(\beta_n r/b)\cos(\beta_n ct/b) \qquad (10)$$

$\beta_n$ : n-th zero of $J_1(x)$

Coefficient $c_n$ is given by

$$(1+\nu)\alpha T_0 c_n = -\frac{2}{J_0^2(\beta_n)}\int_0^1 t\bar{u}(bt)J_1(\beta_n t)dt \qquad (11)$$

Corresponding stresses are

$$\sigma_r = \frac{\alpha T_0 E}{1-\nu}\sum_{n=0}^{\infty} c'_n \left[J_0(z) - (1-\nu)\frac{J_1(z)}{z}\right]_{z=\beta_n r/b} \cos(\beta_n ct/b) \qquad (12)$$

$$\sigma_\theta = \frac{\alpha T_0 E}{1-\nu}\sum_{n=0}^{\infty} c'_n \left[\nu J_0(z) + (1-\nu)\frac{J_1(z)}{z}\right]_{z=\beta_n r/b} \cos(\beta_n ct/b) \qquad (13)$$

$$c'_n = c_n\beta_n/b$$

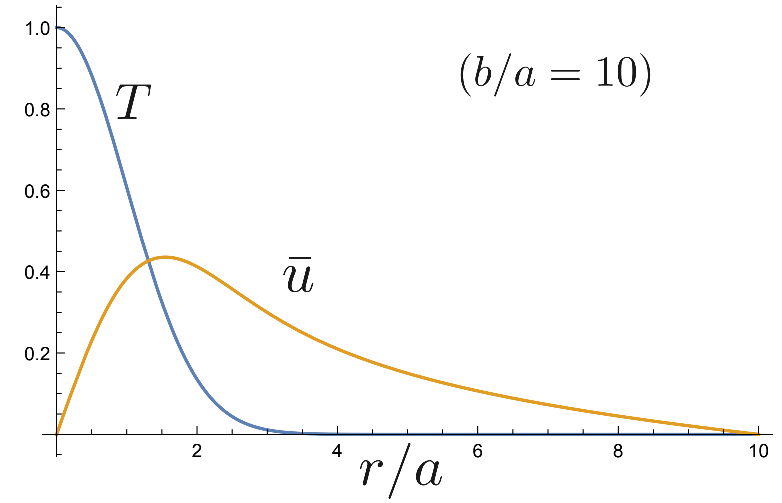
- We denote 'total' stresses as: $p_r \equiv \sigma_r + \bar{\sigma}_r, \quad p_\theta \equiv \sigma_\theta + \bar{\sigma}_\theta$
- At $t=0, \quad \varepsilon_r = \varepsilon_\theta = 0,$

then from (2)(3), $\quad p_r(r,0) = p_\theta(r,0) = -\dfrac{\alpha E T(r)}{1-\nu}$ Initial stress = thermal stress

# Model I. Solution / Gaussian

For a Gaussian beam

$$T(r) = T_0 \exp\left(-\frac{r^2}{2a^2}\right)$$




Static solution:

$$\bar{u} = (1+\nu)\alpha T_0 r \left[f\,(r/a) - f\,(b/a)\right] \tag{14}$$

$$f(x) = (1 - e^{-x^2/2})/x^2$$

then

$$c'_n = -\frac{2}{J_0^2(\beta_n)\lambda^2}\left[1 - e^{-\lambda^2/2} J_0(\beta_n) - \beta_n \int_0^1 e^{-\lambda^2 t^2/2} J_1(\beta_n) dt\right] \tag{15}$$

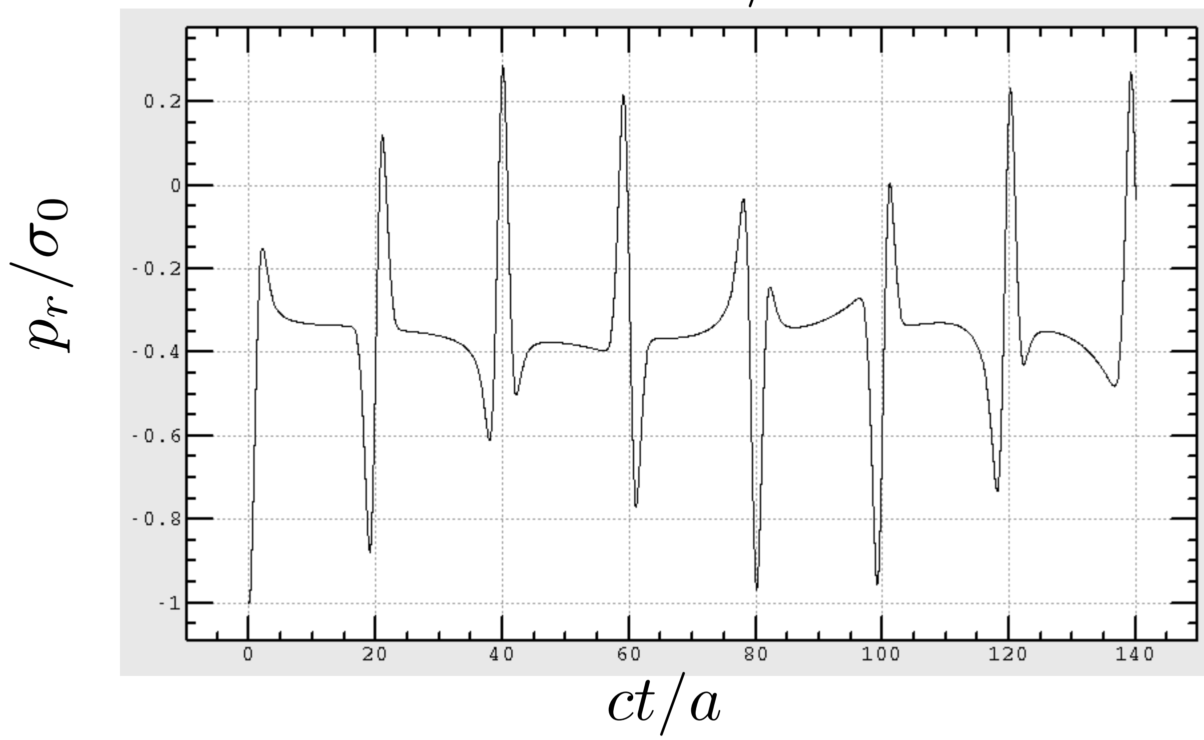
For $\lambda \gg 1$ $\qquad \lambda \equiv b/a$

$$c'_n = -\frac{2}{J_0^2(\beta_n)\lambda^2}\left[e^{-\beta_n^2/2/\lambda^2} - e^{-\lambda^2/2} J_0(\beta_n)\right], \quad (\lambda \gg 1) \tag{16}$$

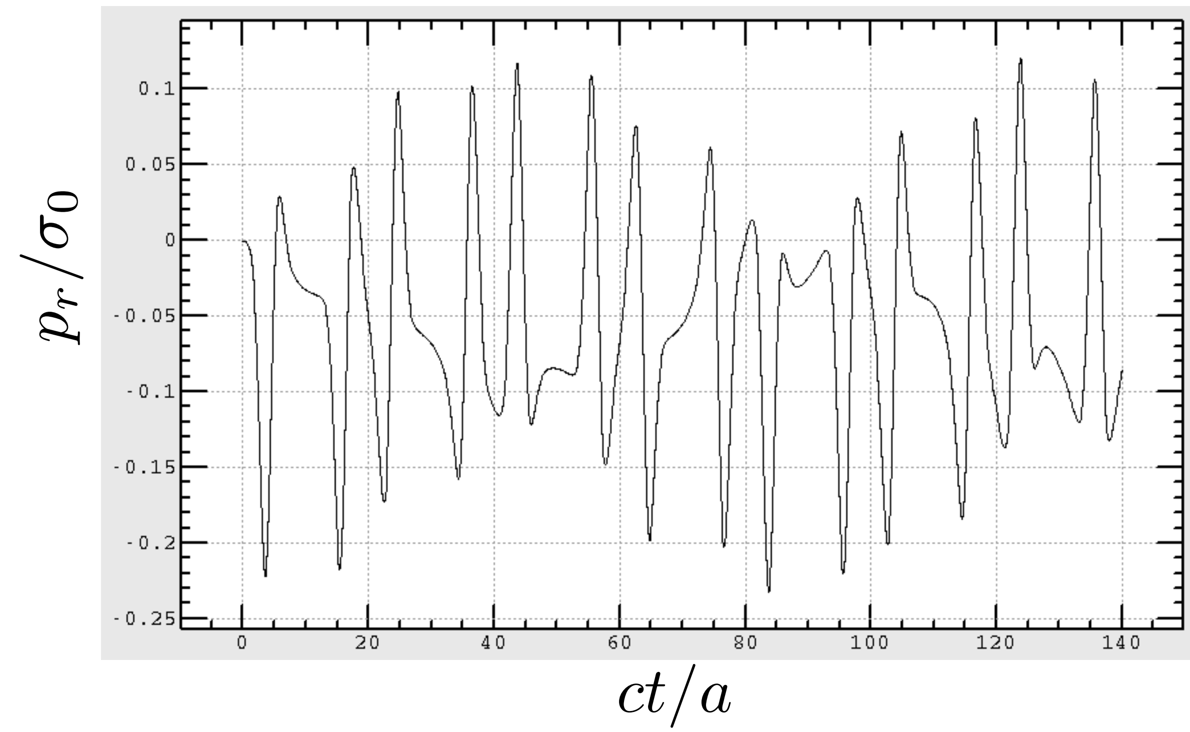
- (16) is accurate in order of $o(e^{-\lambda^2/2})$ .
- Maximum number of terms N in (12)(13) must be such that $\beta_N \gg \lambda$

# Model I.   Solution / Gaussian / Examples

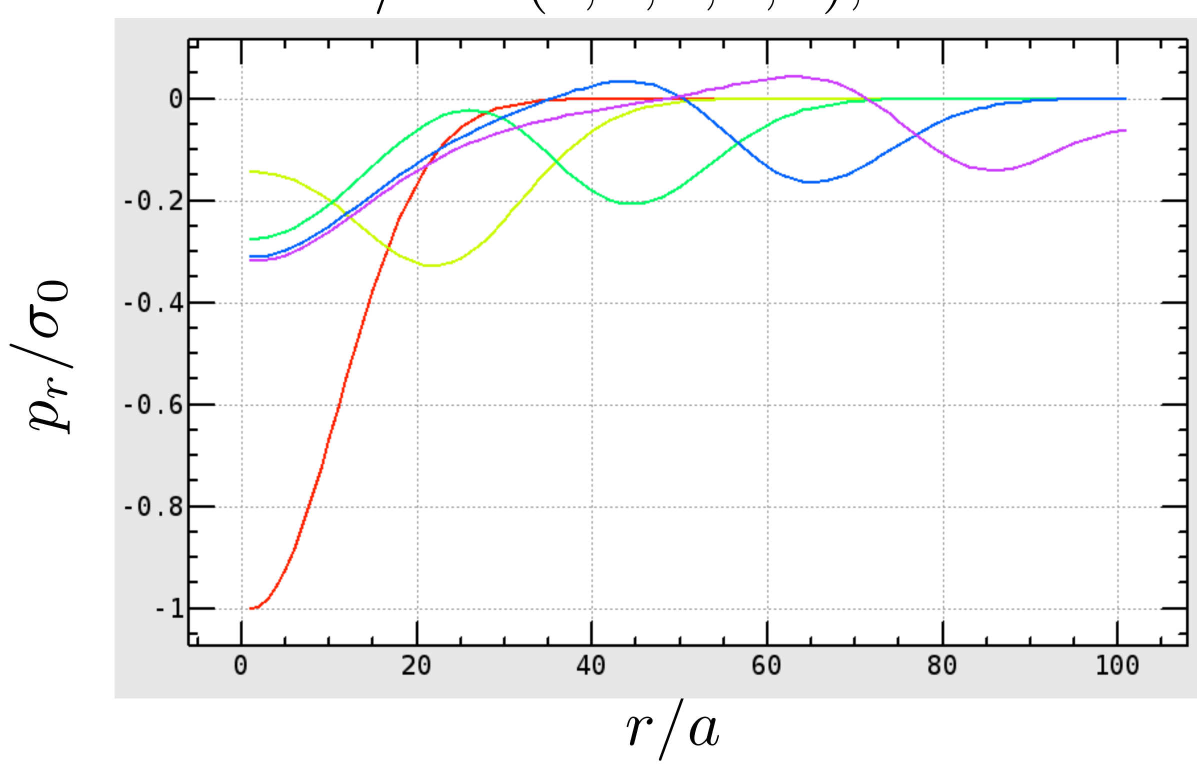
$r = 0 \qquad \lambda = b/a = 10$




$r = 4a \qquad \lambda = b/a = 10$




$ct/a = (0, 2, 4, 6, 8), \quad \lambda = 10$



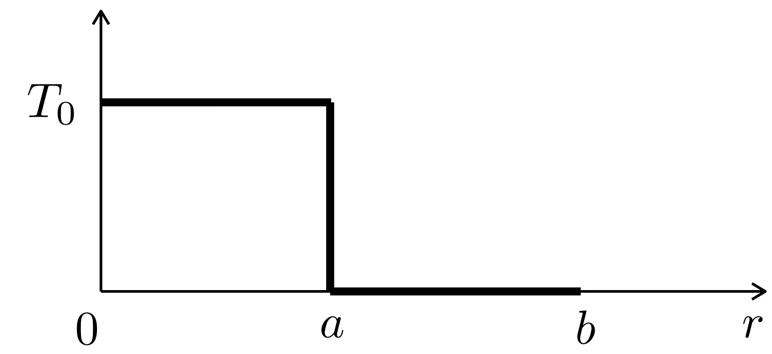

- Multiple echoes.
- ‘Wave packet‘ with a width of beam-size.
- Never exceeds the initial stress.

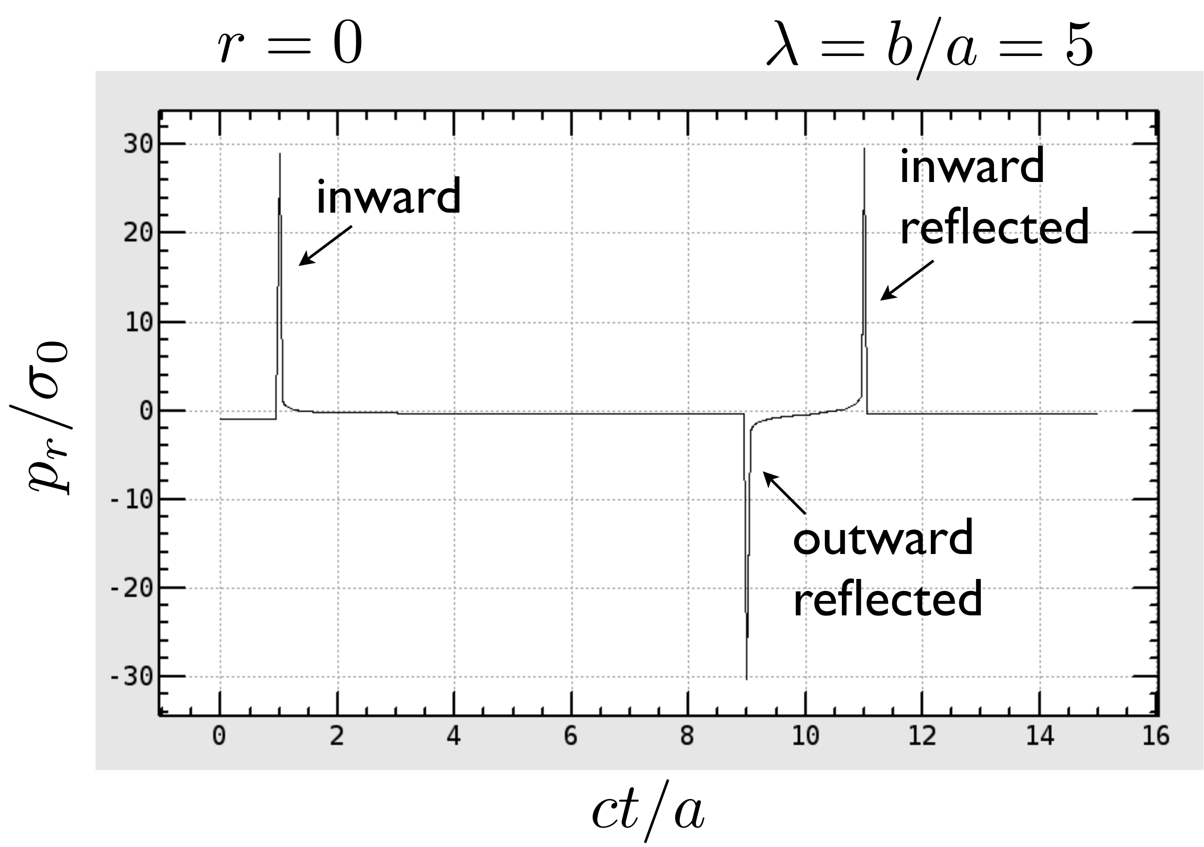
This is a remarkable feature of the Gaussian beam. Rectangular (step function) beam, for example, shows catastrophe (infinite stress) at the origin (P. Sievers, 1974).

# Example: Step-function case

## Example : Step function

$$T(r) = T_0 \quad (0 < r < a)$$
$$= 0 \quad (a < r < b)$$

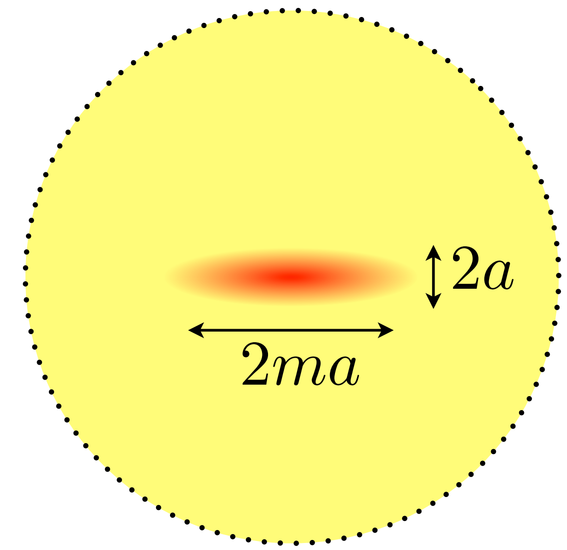



- Multiple echoes .
- Inward wave generated at r=a concentrates at the origin. Catastrophe (infinite stress) at the origin (P. Sievers, 1974).
- The first inward-wave is independent on the boundary conditions.
- Step-wise distribution is dangerous: collimation should be avoided

## Model II.

- Thin disk
- 2-D Gaussian beam with an aspect ratio m

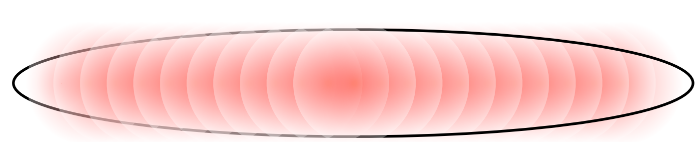


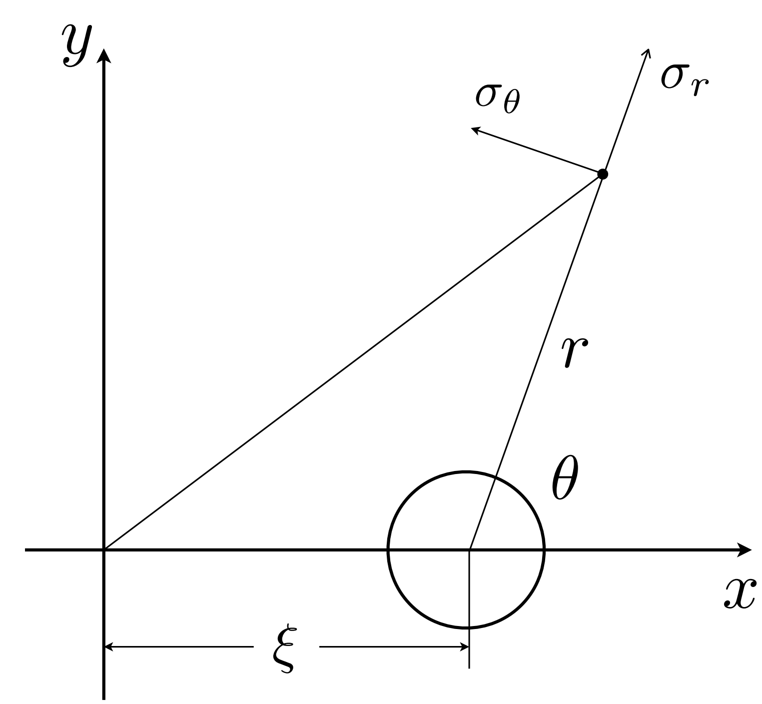

- Wider Gaussian is represented as a superposition of narrower Gaussians

$$e^{-\frac{x^2}{2a_1^2}} = \int_{-\infty}^{\infty} g(\xi) e^{-\frac{(x-\xi)^2}{2a^2}} d\xi \qquad g(\xi) = \frac{a_1}{\sqrt{2\pi} a \sqrt{a_1^2 - a^2}} \exp\left(-\frac{\xi^2}{2(a_1^2 - a^2)}\right)$$

- A 2-D Gaussian is a superposition of round 1-D Gaussians, for which we know the response.

$$e^{-\frac{x^2}{2a_1^2} - \frac{y^2}{2a^2}} = \int_{-\infty}^{\infty} g(\xi) \exp\left(-\frac{(x-\xi)^2 + y^2}{2a^2}\right) d\xi$$





# Model II.

Stresses in Cartesian coordinates

$$p_x = \int_{-\infty}^{\infty} \frac{g(\xi)d\xi}{(x-\xi)^2+y^2}\left[(x-\xi)^2 p_r + y^2 p_\theta\right],$$

$$p_y = \int_{-\infty}^{\infty} \frac{g(\xi)d\xi}{(x-\xi)^2+y^2}\left[y^2 p_r + (x-\xi)^2 p_\theta\right],$$

$$\tau_{xy} = \int_{-\infty}^{\infty} \frac{g(\xi)d\xi}{(x-\xi)^2+y^2}\left[(x-\xi)y(p_r - p_\theta)\right].$$

$(p_r, p_\theta)$

: Stresses in 1-D Gaussian beam

- From symmetry $\tau_{xy} = 0$
- In calculation we used a finite summation with $\Delta\xi = a$, for $m \gg 1$

- This model is correct until the waves reach the boundary, because the geometry of the boundary for each 'micro-bunch' is different each other.

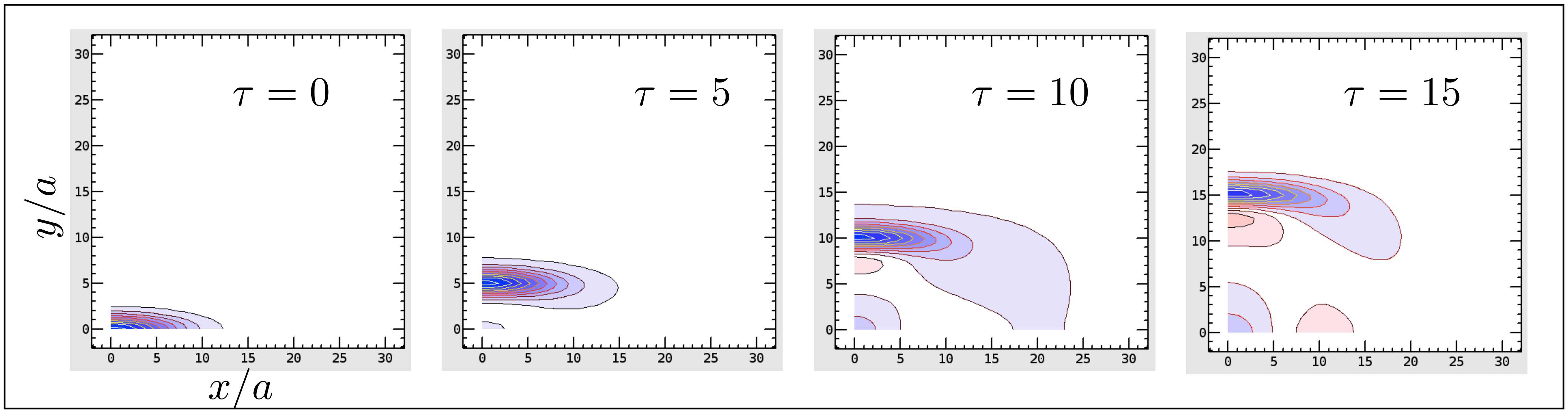
# Model II. Example I

Example I: $m = 5, \quad \lambda = 50$

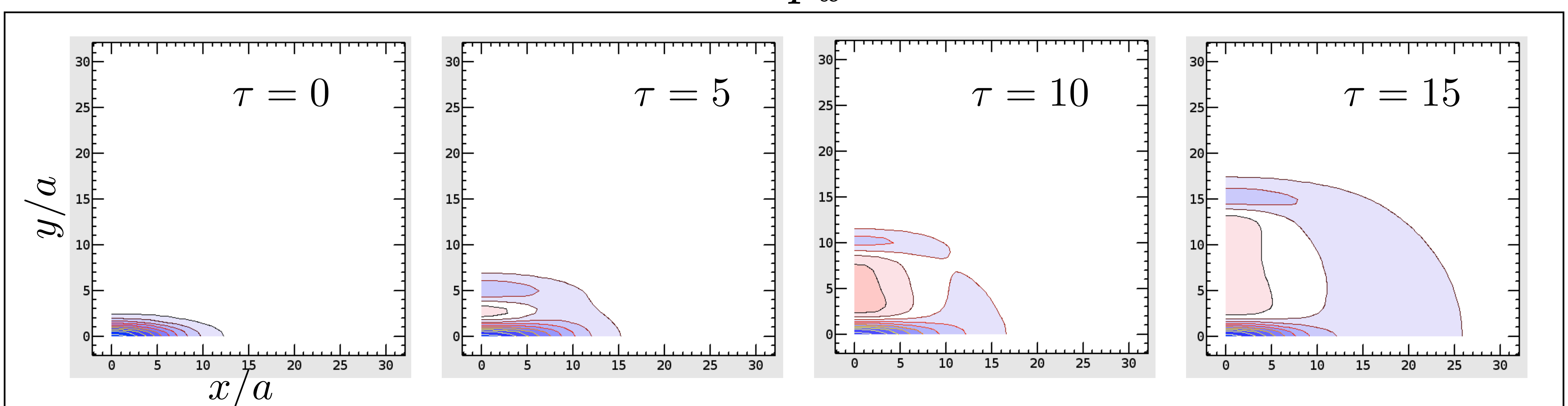
$$T(x,y) = T_0 \exp\left(-\frac{x^2}{2m^2a^2} - \frac{y^2}{2a^2}\right)$$

$\tau \equiv ct/a$

$p_y$

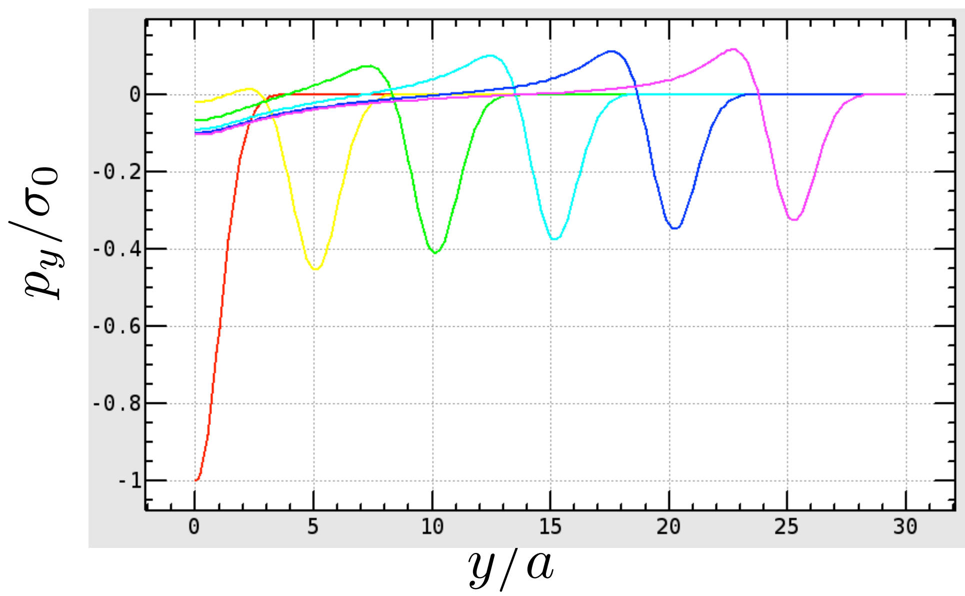



$p_x$

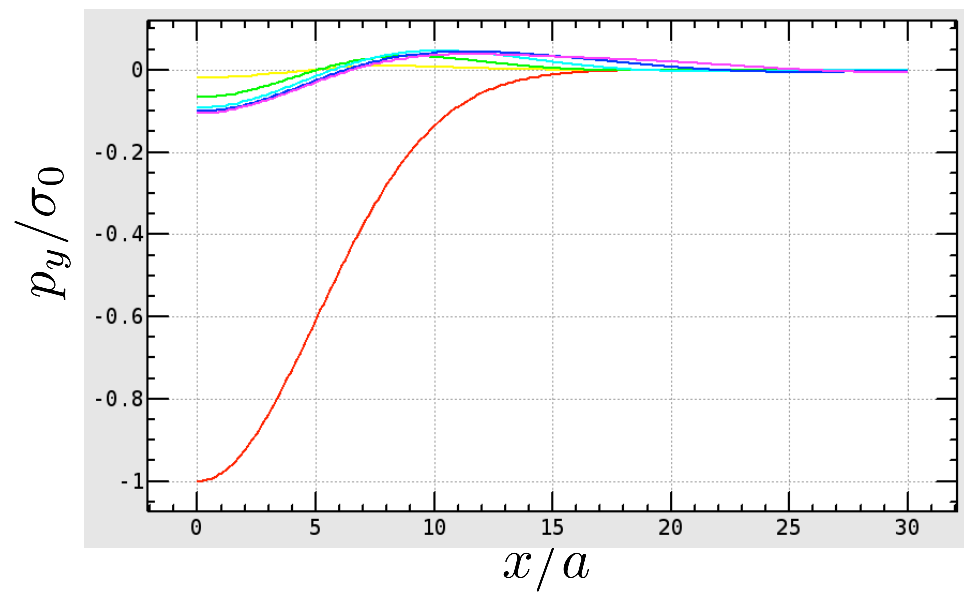


# Model II.   Example I

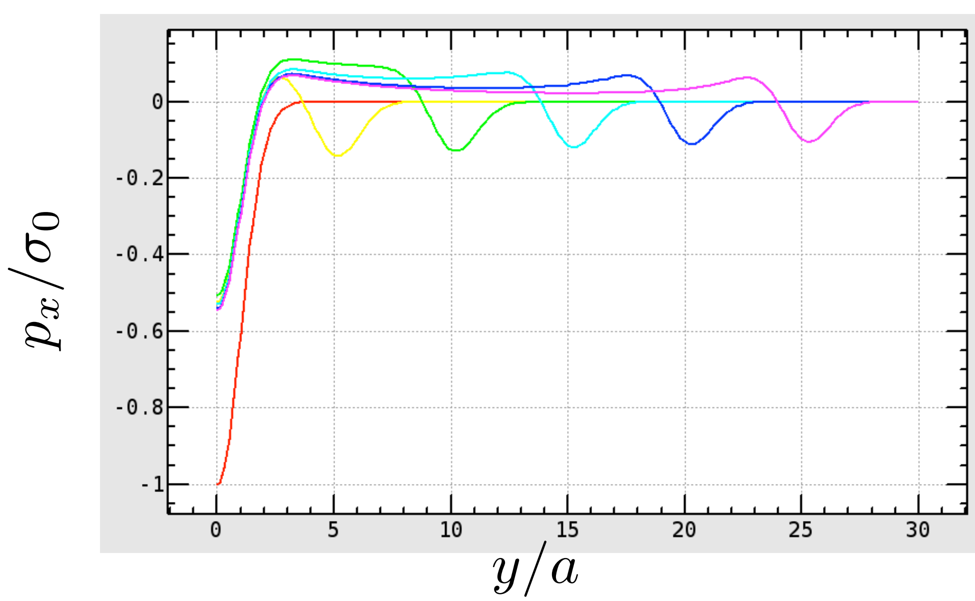
Py on y-axis   $\tau = (0, 5, 10, 15, 20, 25)$



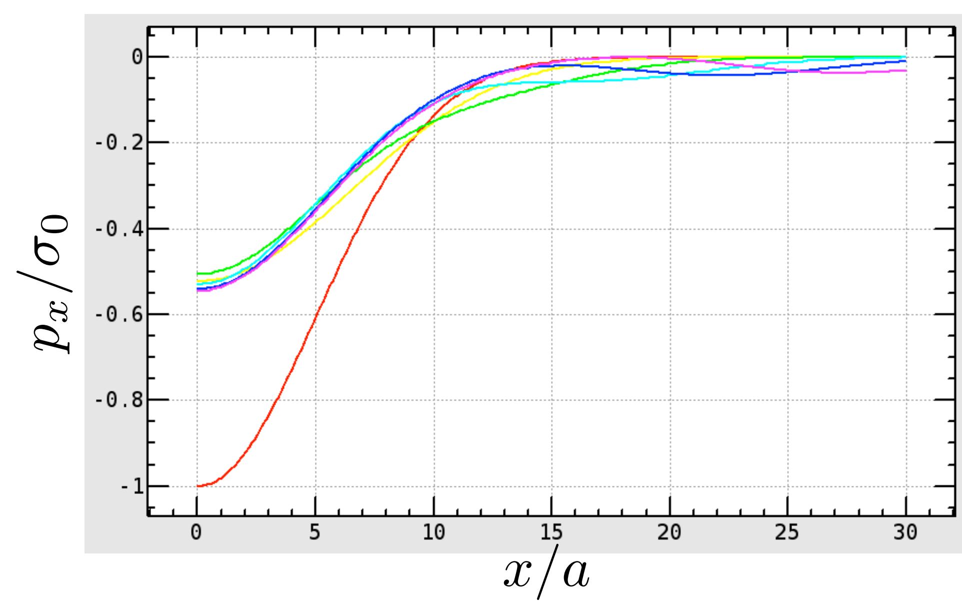

Py on x-axis   $\tau = (0, 5, 10, 15, 20, 25)$




Px on y-axis   $\tau = (0, 5, 10, 15, 20, 25)$




Px on x-axis   $\tau = (0, 5, 10, 15, 20, 25)$

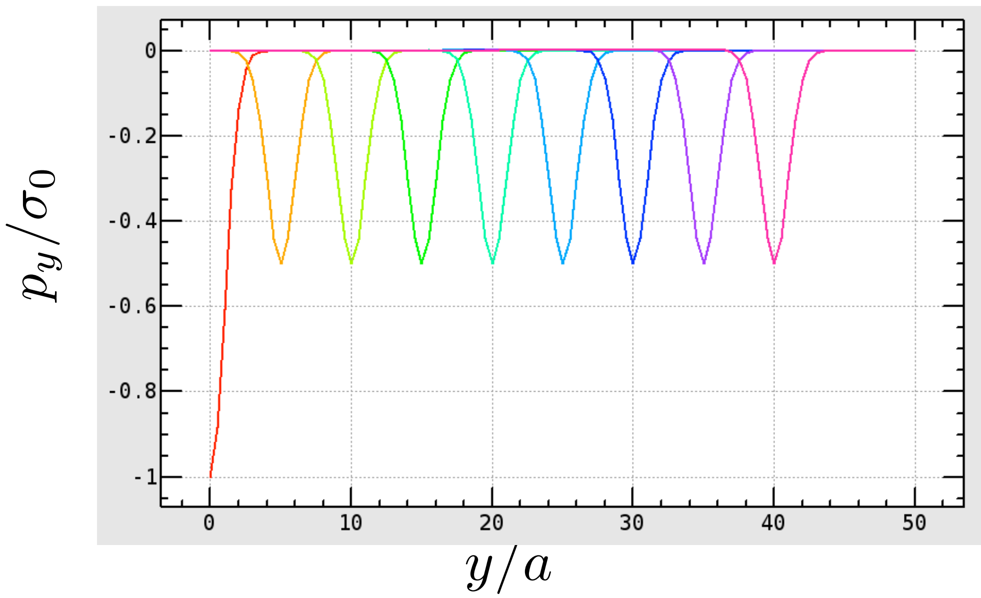



- Py propagates in almost one-dimensional manner
- Px stays longer time around the origin
- Waves along x-axis is very small

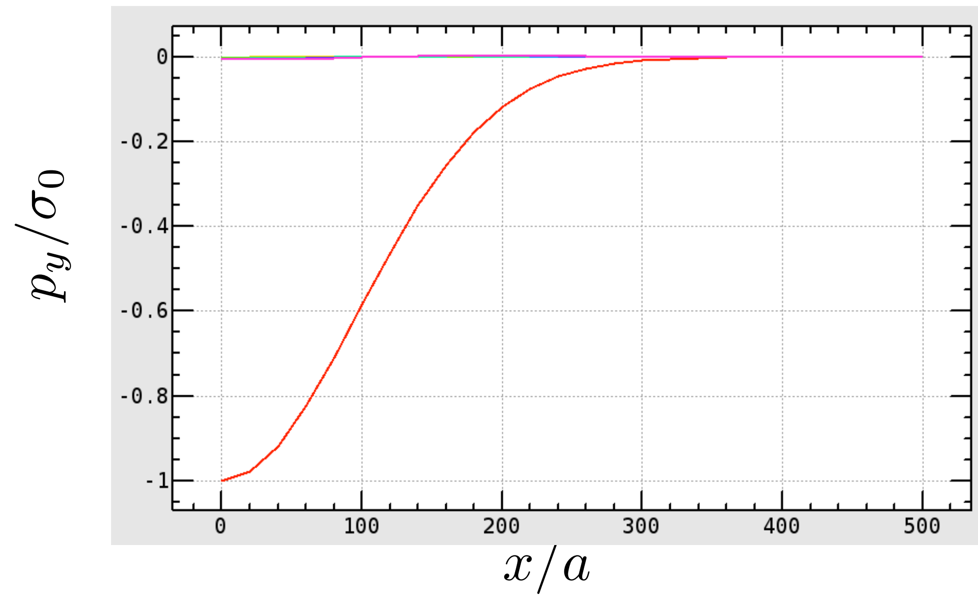
# Model II.  SLER single bunch

Example 2: $m = \sqrt{2} \times 68.3$ (Actual beams of SuperKEKB LER, w/ tilted window)

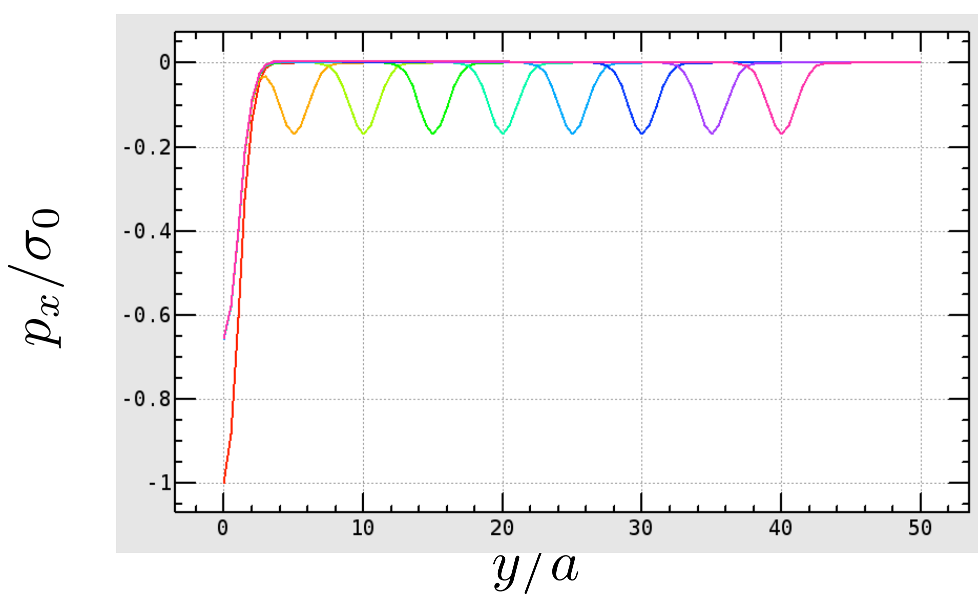
### Py on y-axis $\tau = (0 \sim 40)$



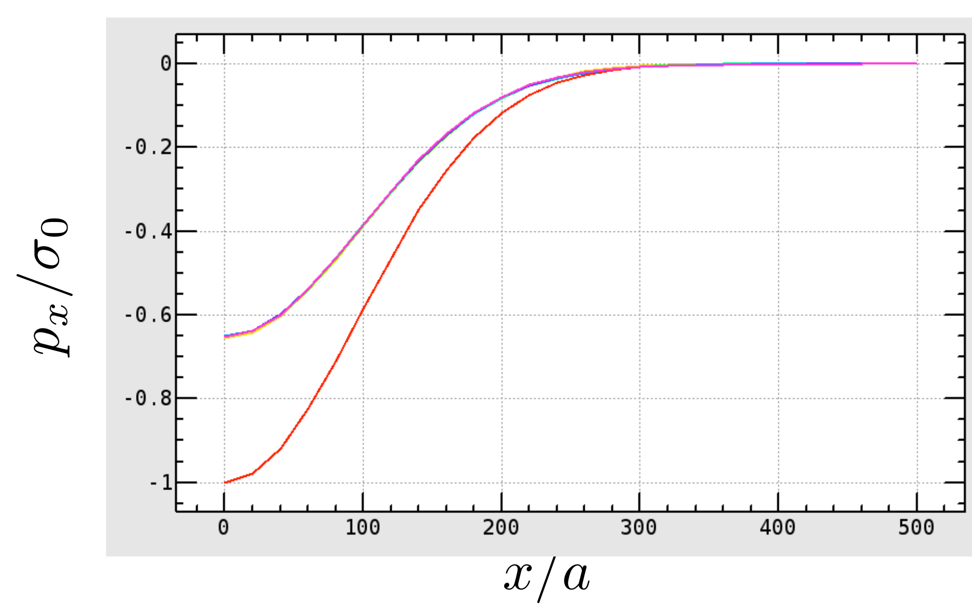

### Py on x-axis $\tau = (0 \sim 300)$




### Px on y-axis $\tau = (0 \sim 40)$




### Px on x-axis $\tau = (0 \sim 300)$

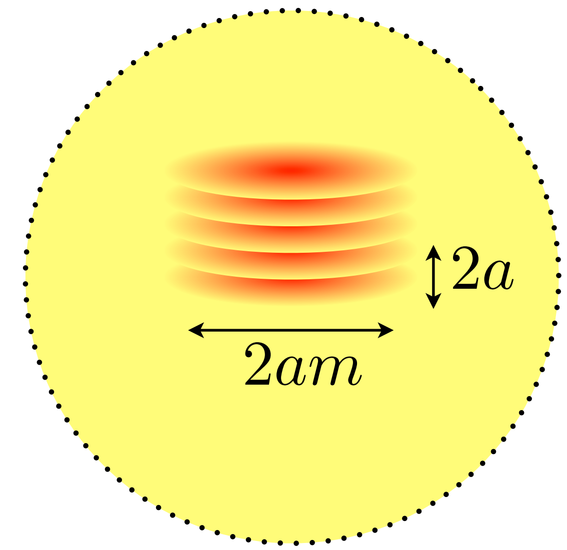



- Py propagates in almost one-dimensional manner
- Px stays longer time around the origin
- Waves along x-axis is very small

$p_y \sim 0.5\ \sigma_0$

$p_x \sim 0.65\ \sigma_0$

# Model III. SLER multi-bunch

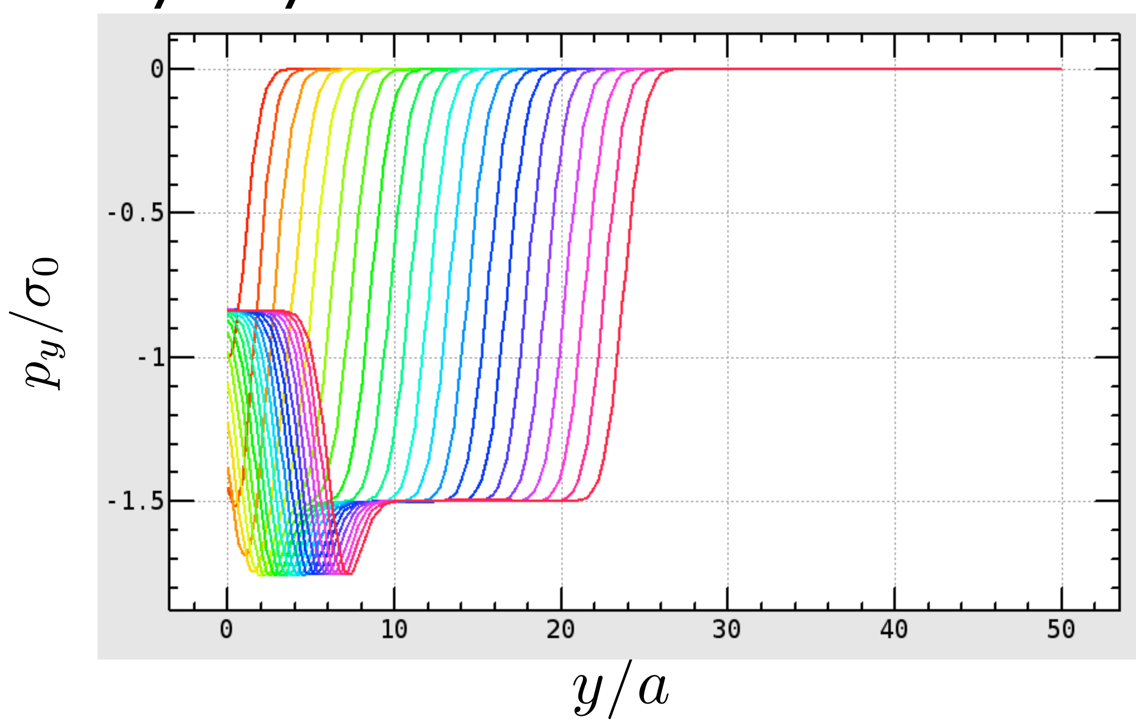



## Model III.

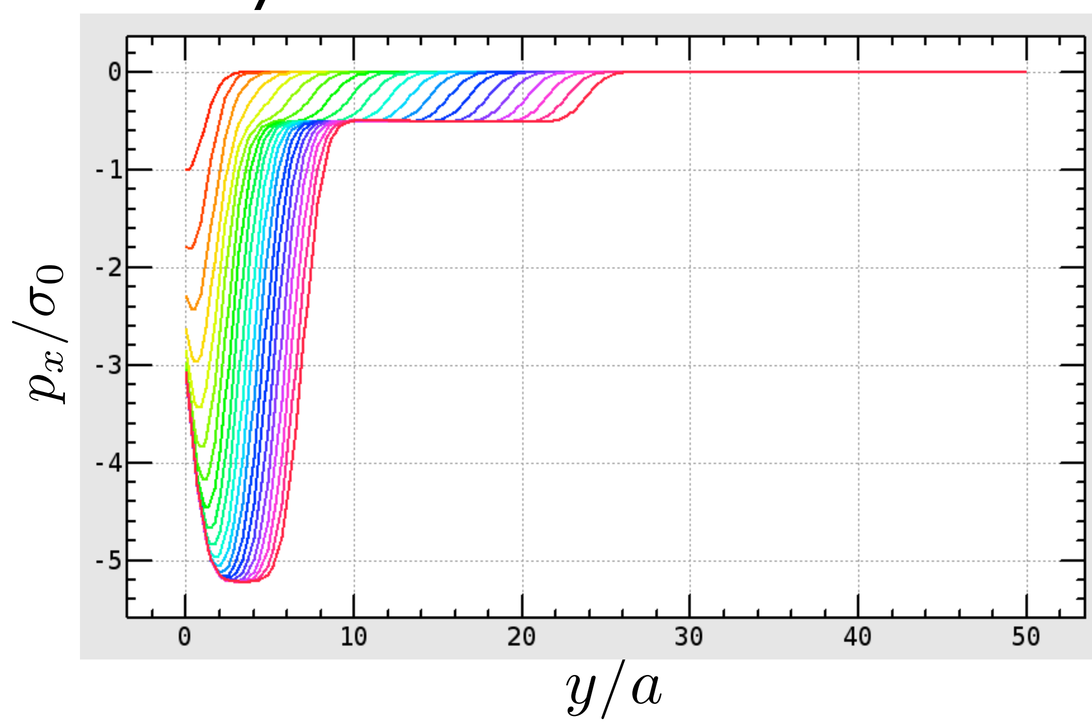
- Thin disk
- 2-D Gaussian beam
- Multi-bunch, scanned vertically

  - Results of Model II are overlaid with shifted vertical positions and shifted time.

Py on y-axis 1 ~ 20 bunches




Px on y-axis 1 ~ 20 bunches



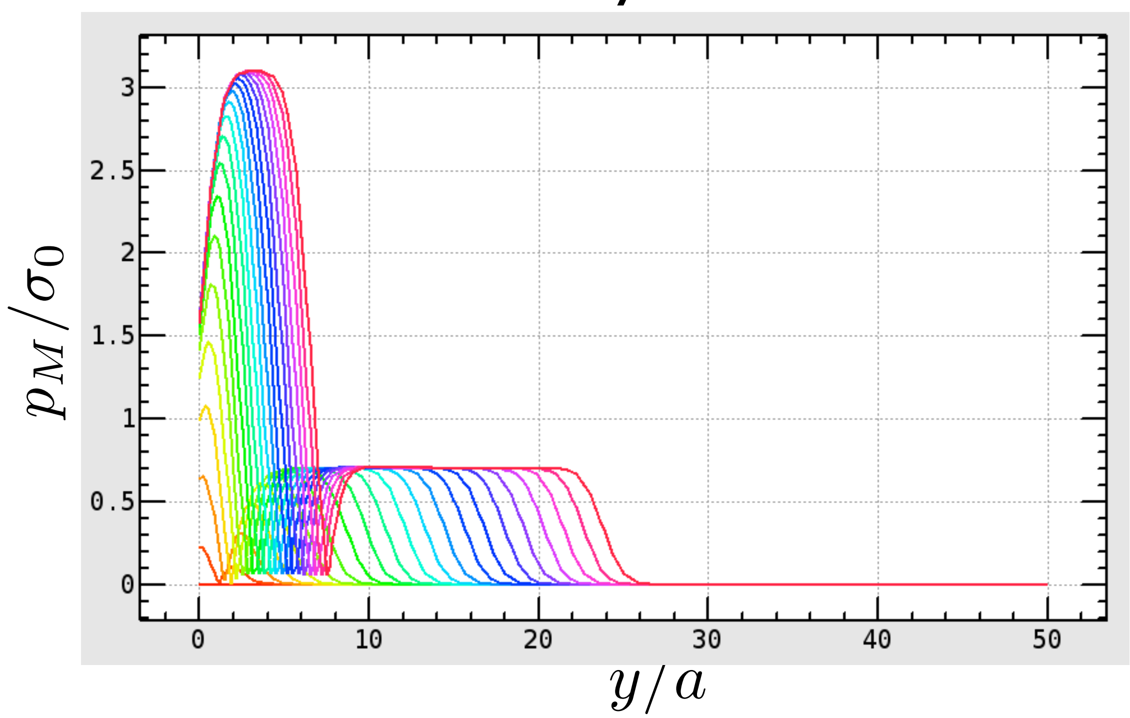

- Maximum stress is $5.2\ \sigma_0 = 274\ \mathrm{MPa}$
- $$p_{y,max} \simeq \frac{\sqrt{2\pi}\sigma_y}{(c\Delta t - \Delta y)} 0.5\,\sigma_0 = 1.5\,\sigma_0 \qquad p_{x,max} \simeq \frac{\sqrt{2\pi}\sigma_y}{\Delta y} 0.65\,\sigma_0 = 4.9\,\sigma_0$$

# Model III. Multi-bunch, von Meses stress

- The "von Mises stress" is responsible to yield of materials

$$p_M^2 = \left[(p_x - p_y)^2 + (p_y - p_z)^2 + (p_z - p_x)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)\right]/2$$

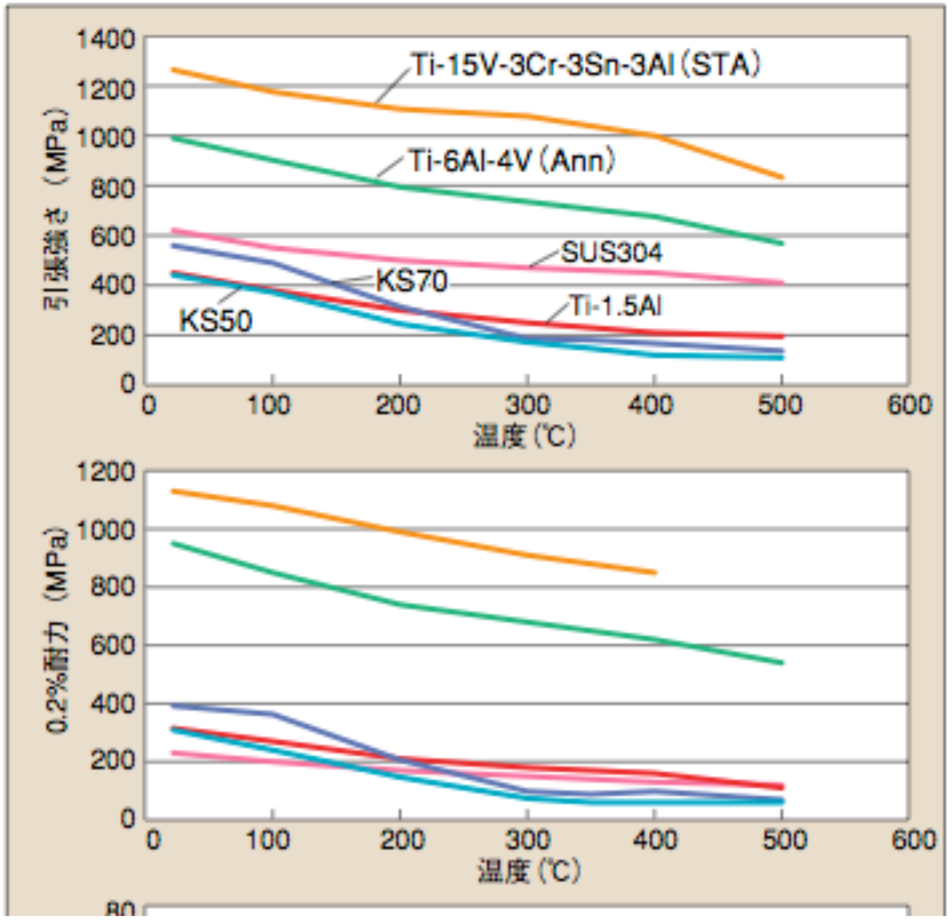
Von Mises stress on y-axis 1 ~ 20 bunches




- Maximum Von Mises stress is $3.2\,\sigma_0 = 168$ MPa
- Taking into account possible overlap with reflected waves, maximum Von Mises stress is expected to be $\sqrt{(|p_{x,max}| + |p_{y,max}|)^2/2} = 4.9\,\sigma_0 = 258$ MPa
- This is larger than the yield strength of pure Ti.

<u>Model III. Multi-bunch, von Meses stress</u>

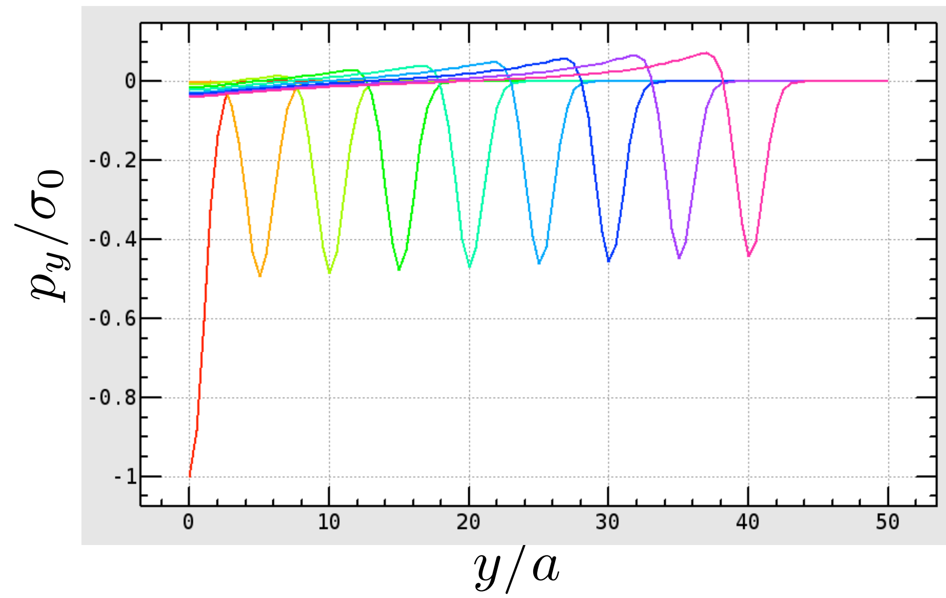
- Ti alloy (Ti-6Al-4V) may be a candidate.



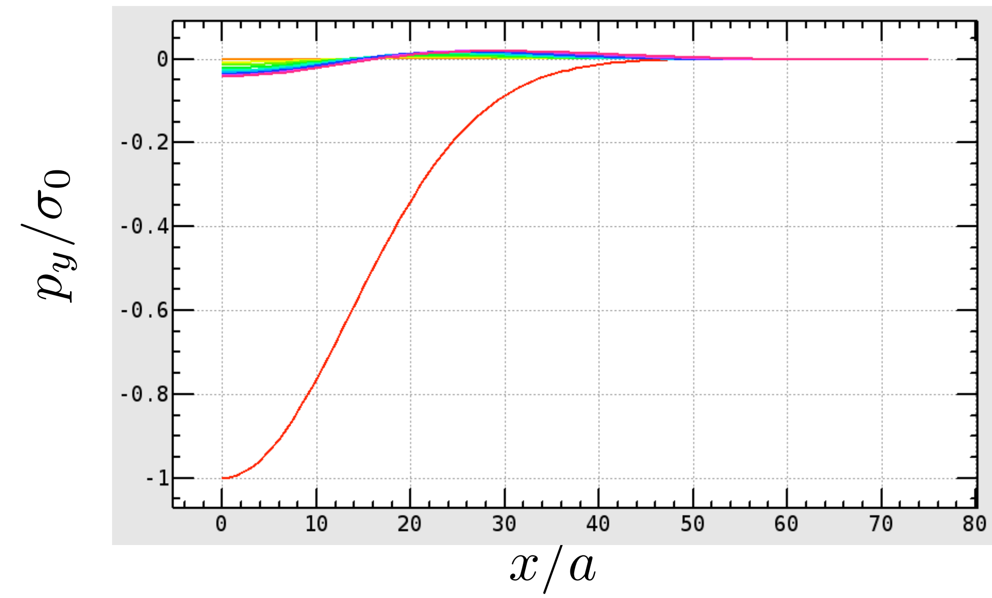
# KEKB, LER single bunch

Example 3: $m = \sqrt{2} \times 9.6$ (Actual beams of KEKB, w/ tilted window)

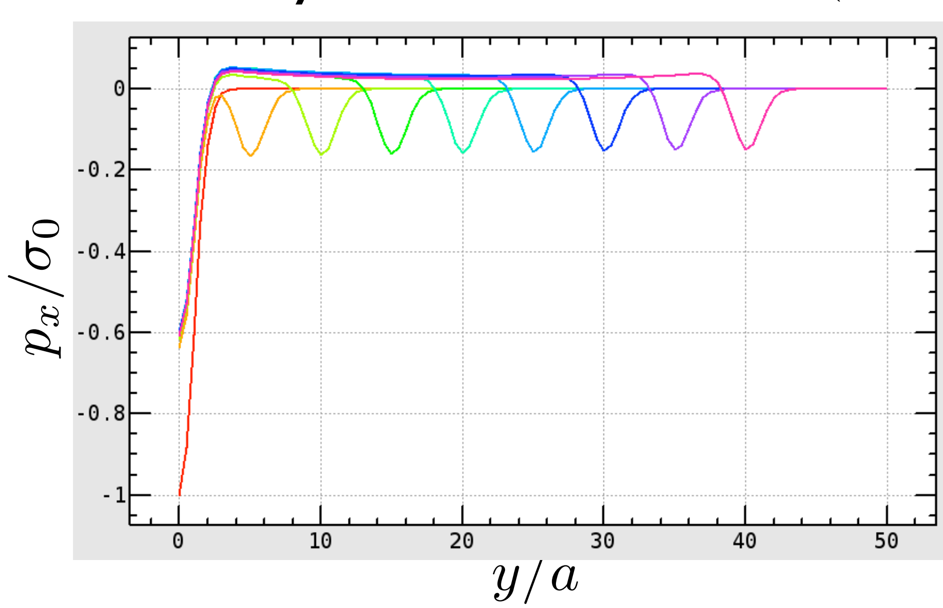
Py on y-axis $\tau = (0 \sim 40)$



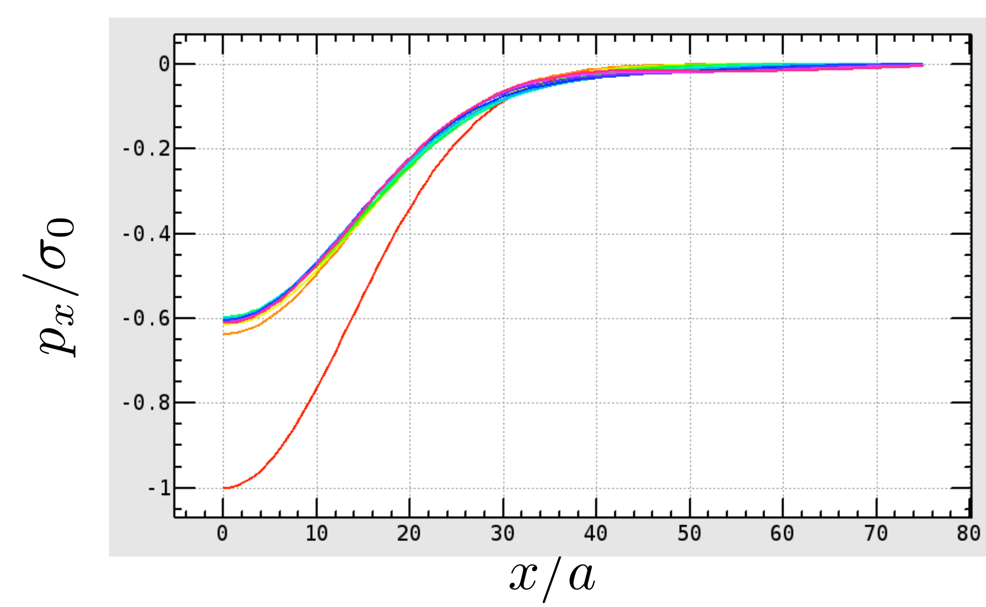

Py on x-axis $\tau = (0 \sim 50)$




Px on y-axis $\tau = (0 \sim 40)$




Px on x-axis $\tau = (0 \sim 50)$



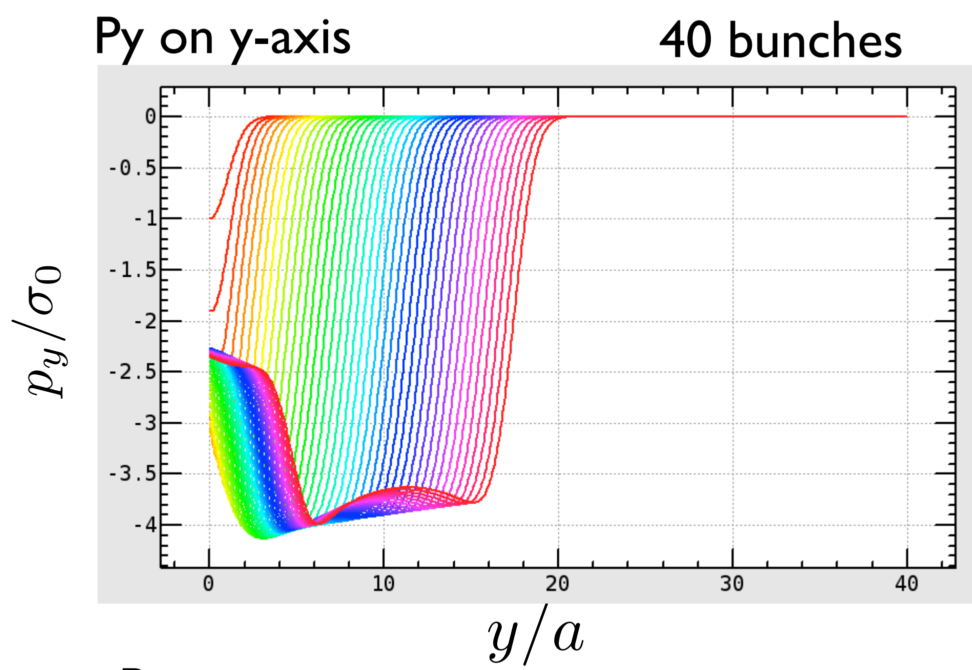

- Py propagates in almost one-dimensional manner $p_y \sim 0.45\ \sigma_0$
- Px stays for longer time around the origin $p_x \sim 0.6\ \sigma_0$
- Waves along x-axis is very small

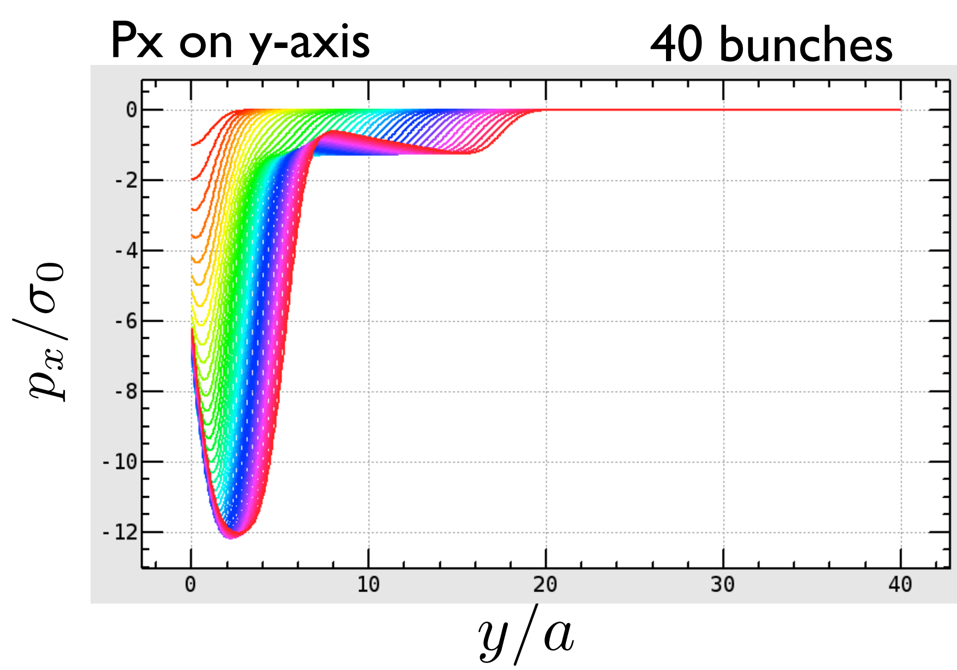
# KEKB, LER multi-bunch

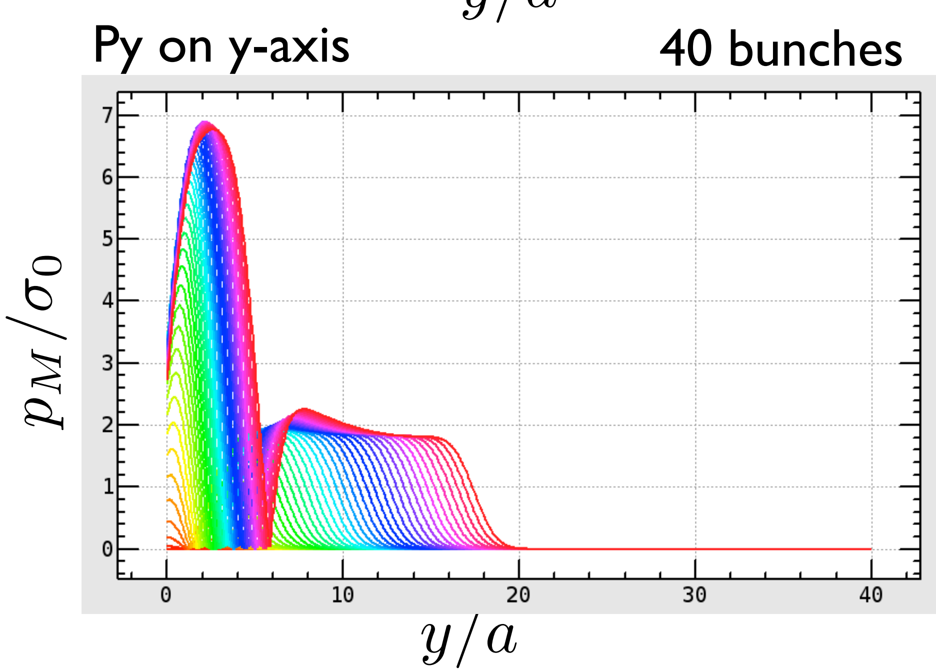
$\sigma_y = 73\ \mu\mathrm{m}, \quad \mathrm{c}\Delta\mathrm{t} = 31.5\ \mu\mathrm{m}, \quad \Delta\mathrm{y} = 9\ \mu\mathrm{m}$

$\sigma_0 = 16.2\ \mathrm{MPa}$

$$p_{y,max} \simeq \frac{\sqrt{2\pi}\sigma_y}{(c\Delta t - \Delta y)} 0.45\,\sigma_0 = 3.72\,\sigma_0 \qquad p_{x,max} \simeq \frac{\sqrt{2\pi}\sigma_y}{\Delta y} 0.57\,\sigma_0 = 11.6\,\sigma_0$$




- Maximum stress : $11.6\,\sigma_0 = 188\ \mathrm{MPa}$
- Max. von Meses stress : $6.9\,\sigma_0 = 112\ \mathrm{MPa}$
- Max. von Meses with reflections: $180\ \mathrm{MPa}$
- Maximum stress was very close to the yield strength.

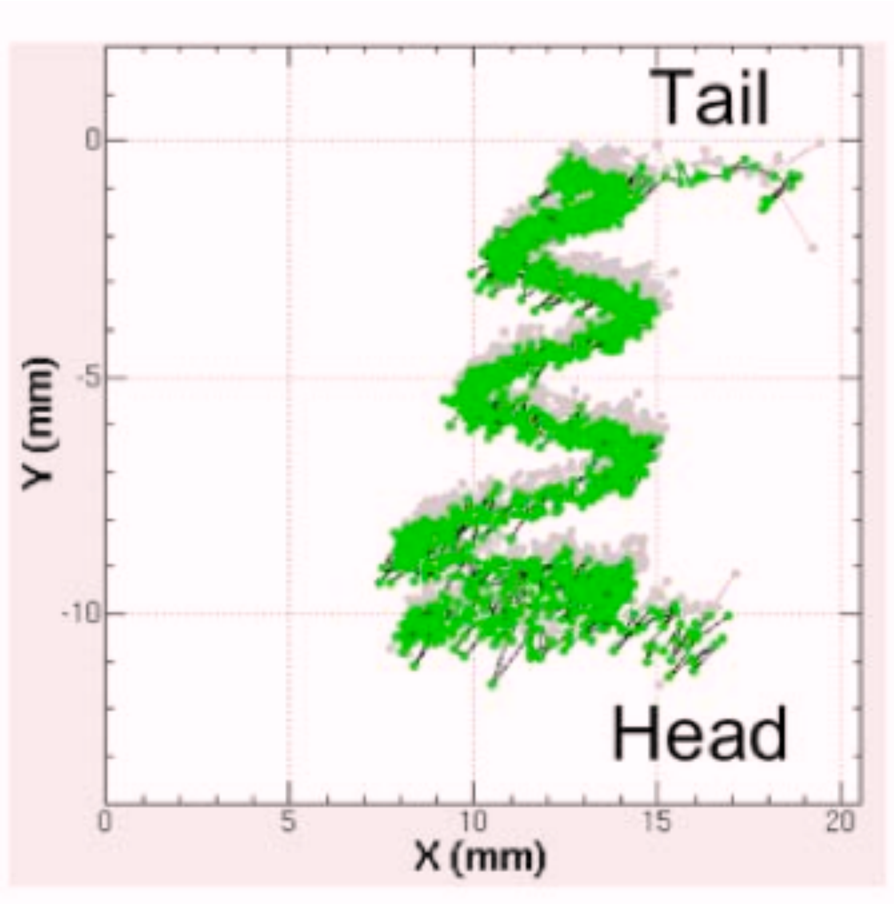
(Ignorance is bliss! 知らぬが仏)

## Summary

- SuperKEKB abort-system uses maneuvers of the vertical scan and horizontal enlargement to effectively reduce the power dissipation in the extraction window.
- We have developed an analytical method applicable to the dynamic stresses generated by the two-dimensional Gaussian beam.
- For the Gaussian distribution, the dynamic stress never exceeds the initial thermal stress.
- Maximum von Mises stress is 260 MPa.
- For KEKB case, von Mises stress is estimated as 180 MPa.
- Ti alloy may be a candidate as the window material.

# Spares

Beam trajectory at the beam dump of KEKB HER



# Spares

## Annealing

1. 最大温度　300+25=325℃

- 再結晶温度　~450 ℃

  これを越えて加熱すると結晶粒径が大きくなり、強度が落ちる

  結晶粒径　$\propto$　(加熱時間)$^{1/3}$

$$T_{max} = T_0 \frac{\sqrt{2\pi}\sigma_y}{\Delta y} = \frac{1}{C}\left(\frac{1}{\rho}\frac{dE}{dx}\right)\frac{N}{\sqrt{2\pi}\sigma_x \Delta y}$$

- $\sigma_x \Delta y$　は再結晶温度に対して1.4 倍の余裕がある

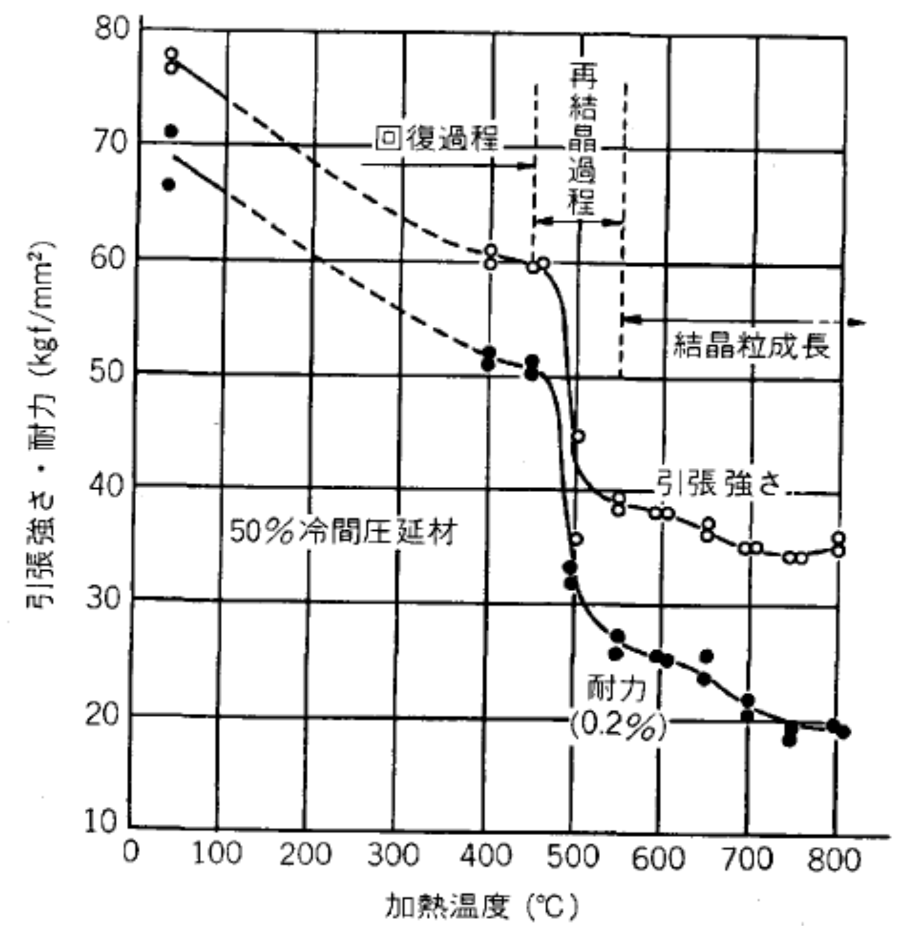
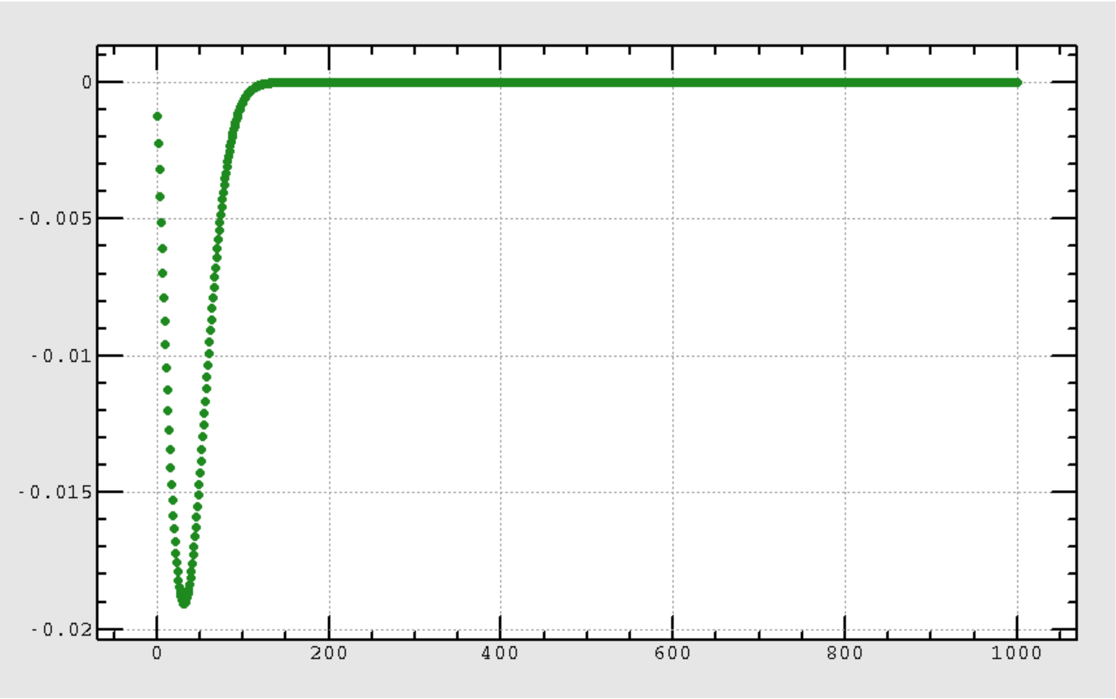



図 1　純チタンの軟化曲線

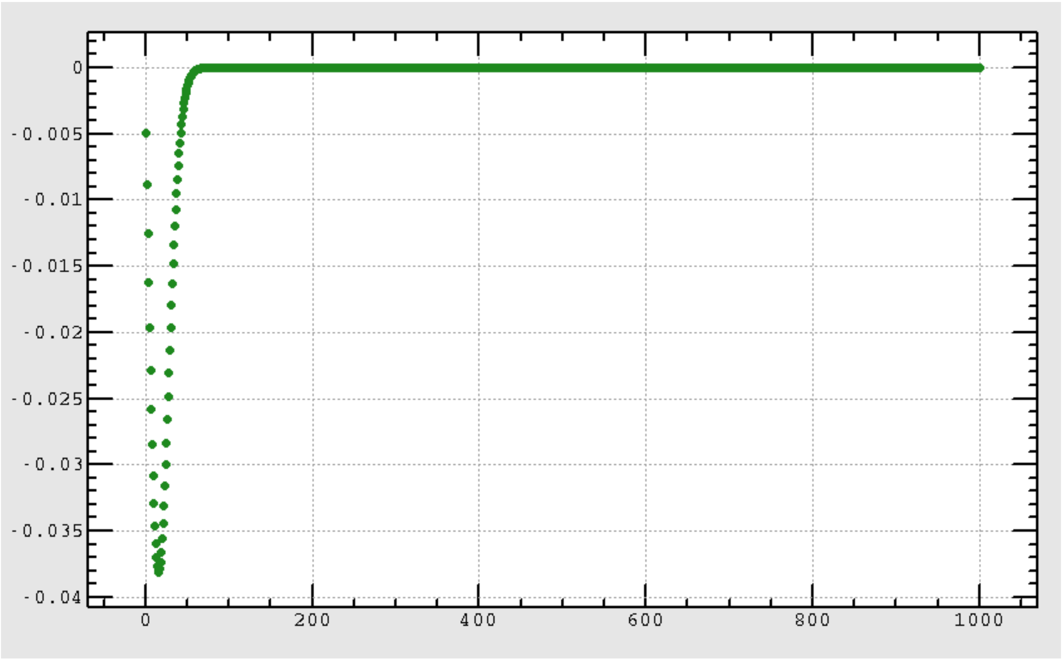
西村孝、日本鐵鋼協會々誌 70(15),1898-1905,1984-11-01

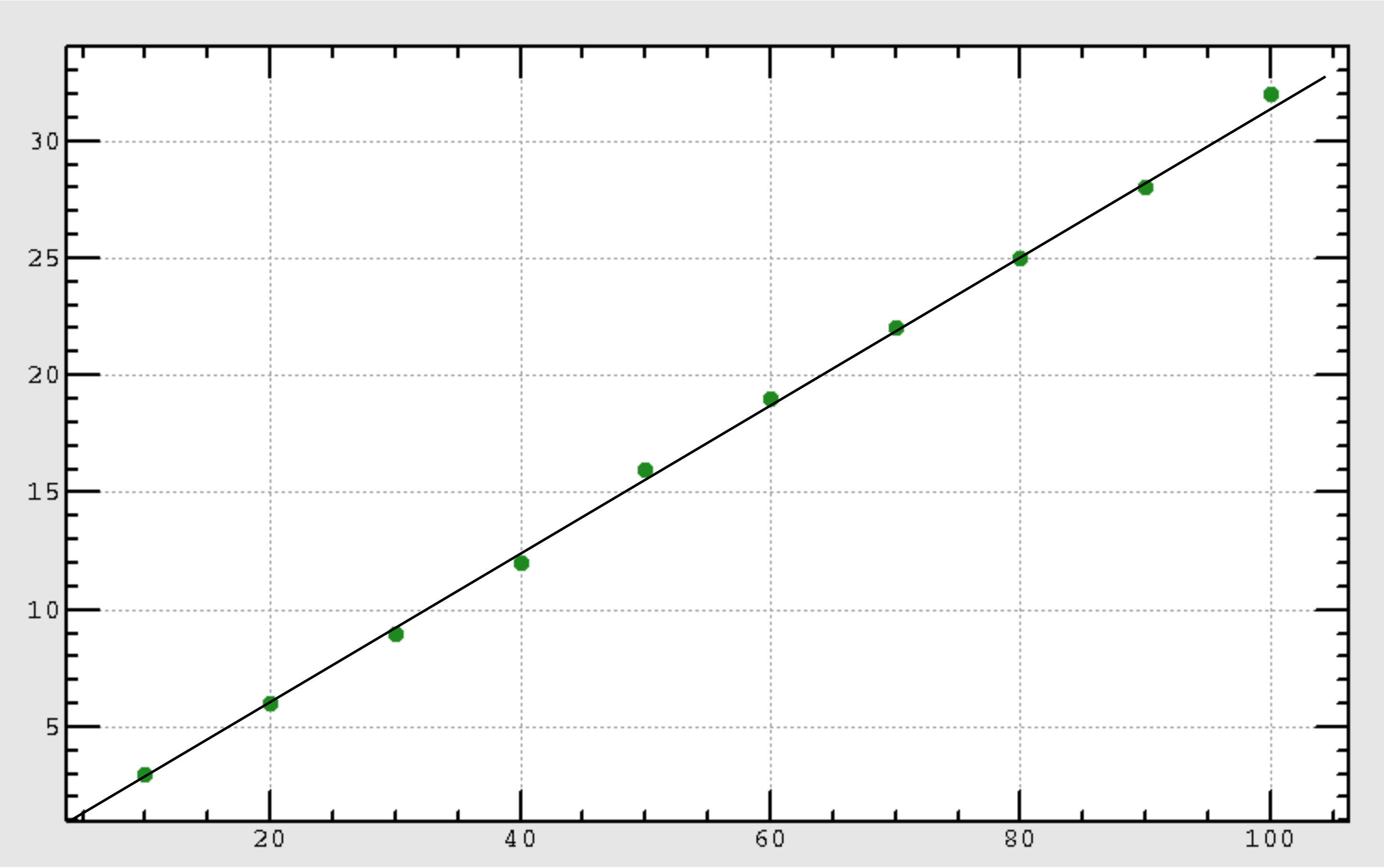
# Spares

c'(n) lambda=100




c'(n) lambda=50




$\lambda$ vs n that maximizes |c'(n)|

$$n_{max} \simeq 5/16\lambda$$

# Spares

$$c'_n = -\frac{2}{J_0^2(\beta_n)\lambda^2}\left[e^{-\beta_n^2/2/\lambda^2} - e^{-\lambda^2/2}J_0(\beta_n)\right], \quad (\lambda \gg 1)$$

は、さらに

$$c'_n = -\frac{2}{J_0^2(\beta_n)\lambda}e^{-\beta_n^2/2/\lambda^2}$$

と近似できる。$\beta_n \simeq \pi(n+1/4)$ および漸近式

$$J_0(z) \sim \sqrt{\frac{2}{\pi z}}\cos(z-\pi/4)$$

から、

$$\frac{1}{J_0^2(\beta_n)} \sim \frac{\pi}{2}\beta_n$$

が成立する。したがって

$$c'_n \simeq -\pi\frac{\beta_n}{\lambda^2}e^{-\beta_n^2/2/\lambda^2}$$

これから、$c'_n$ を最大にする $\beta_n$ は

$$\beta_{n_{\max}} = \lambda$$

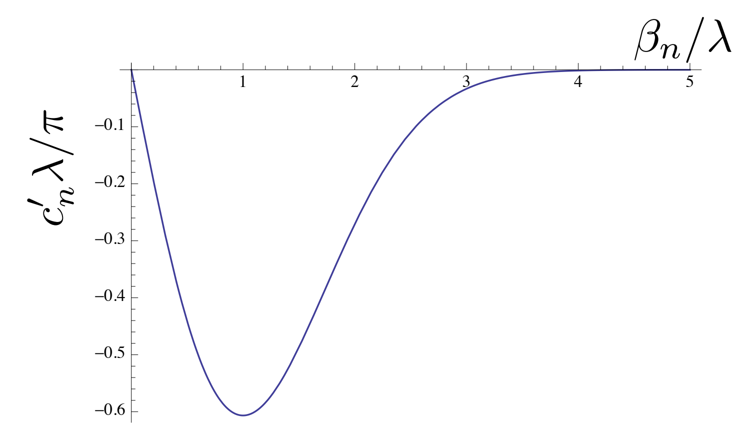
よって、波の主成分周波数は

$$\omega_{\mathrm{main}} = \beta_{n_{\max}}c/b = \lambda(c/b) = c/a$$

波長 $\Lambda$ は

$$\Lambda = 2\pi a$$

となる。$c'_n$ を $\beta_n/\lambda$ の関数として描くと次図の様になる。



# Spares

この $c'_n$ の分布に関して

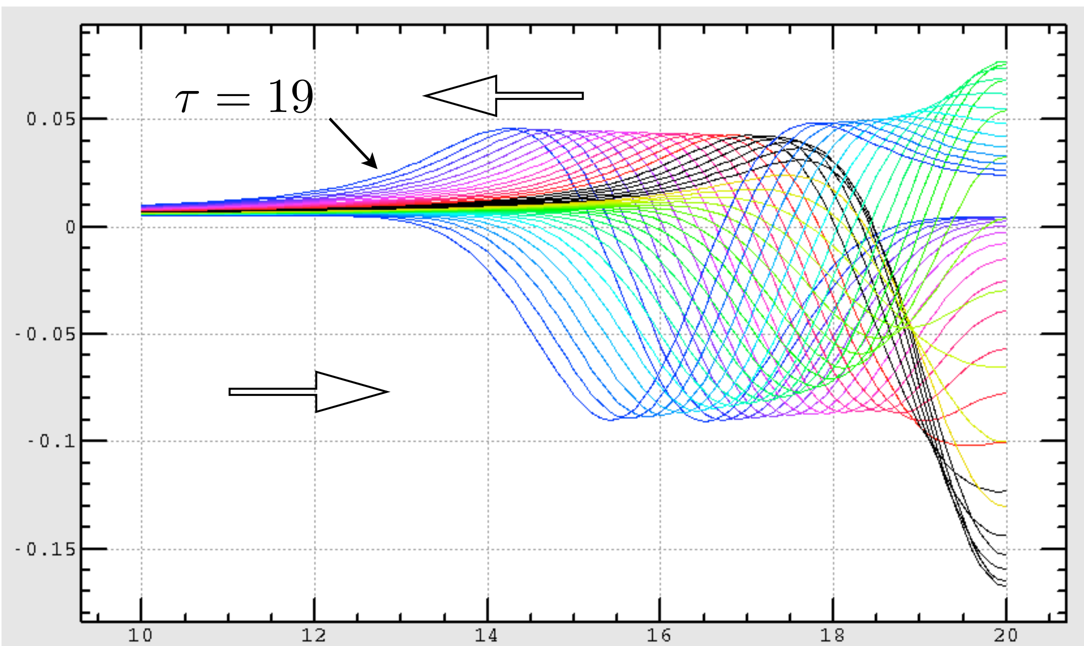
$$\langle \beta_n \rangle = \sqrt{\pi/2}\,\lambda \qquad \langle \beta_n^2 \rangle = 2\,\lambda^2$$

$$\Delta\beta_n \equiv \sqrt{\langle \beta_n^2 \rangle - \langle \beta_n \rangle^2} = \sqrt{2 - \pi/2}\,\lambda$$

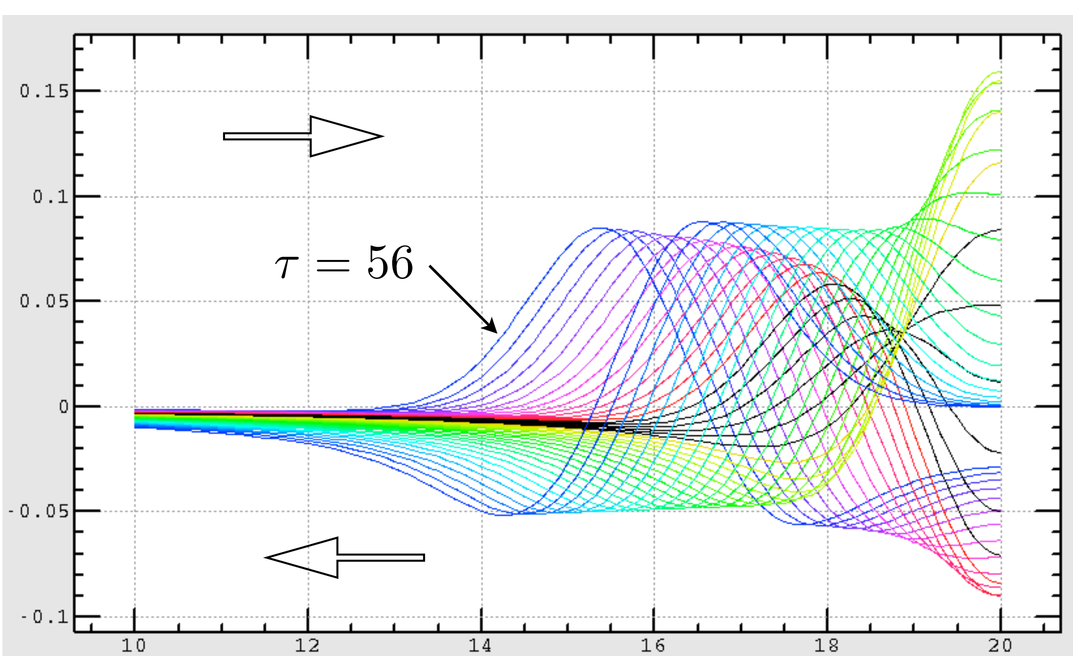
# Spares

## 境界での折り返しの様子

$$\lambda = 20, \quad \tau = \{16, 24, .2\}$$

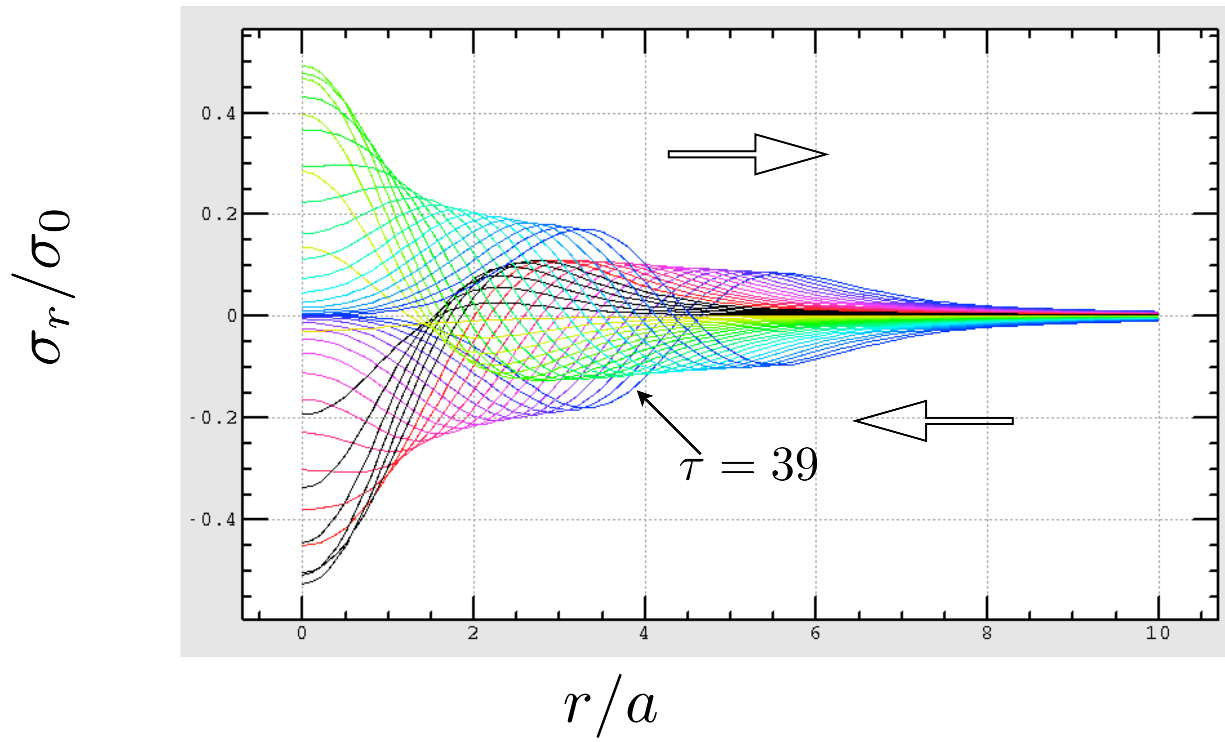


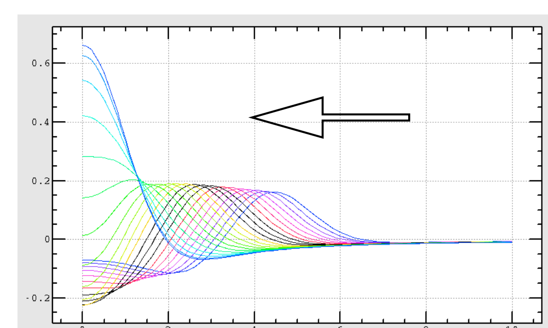

$$\lambda = 20, \quad \tau = \{56, 64, .2\}$$



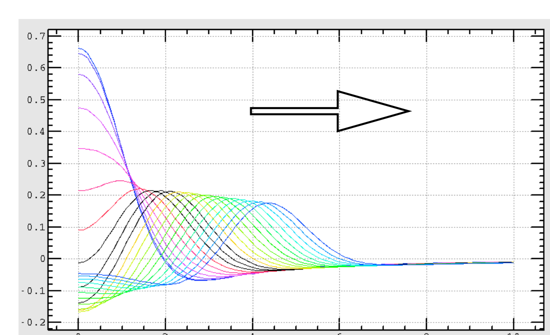

## 原点での折り返しの様子

$$\lambda = 20, \quad \tau = \{36, 44, .2\}$$



$$\tau = \{76, 80, .2\}$$



$$\tau = \{80, 84, .2\}$$