BT Emittance Blowup

Systematic Effects in Emittance Measurements

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29th KEKB Accelerator Review Committee

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- Magnet healthiness check
- Reconsideration and consequences of emittance blowup mechanism
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  - Archived data analysis
  - BPM healthiness survey
  - BPM response imbalance scenario
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  - Summary and study proposal

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BT emittance history

N. Iida, *et al.*

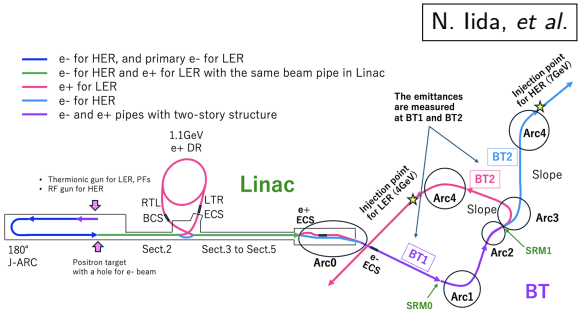


Figure 3: Layout of the SuperKEKB injector complex. Emittances of both e^+ and e^- beams are measured at "BT1" using four wire scanners (WSs) and at "BT2" using an optical transition radiation (OTR) screen and two quadrupoles with the "Q-scan" method. Two synchrotron radiation monitors (SRM0 and SRM1) continuously measure the beam size non-destructively during injection.

N. Iida, *et al.*

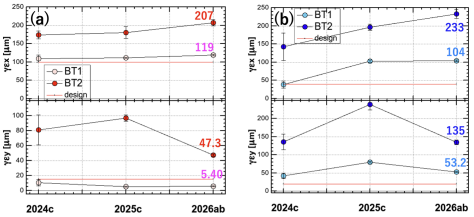


Figure 4: Average emittances for the 2024c, 2025c, and 2026ab operations: (a) e^+ and (b) e^-. Upper and lower plots show the horizontal and vertical emittances at BT1 and BT2, respectively. Red lines indicate the ring design values. The bunch charge was 3 nC for e^+ and 2 nC for e^-.

Table 2: The Courant-Snyder invariants of the injected beam and the measured scceptance of the ring

	CSI	Injected beam		Ring acc.
HER	$2J_x$ [μm]	1.01	<	1.4
	$2J_y$ [μm]	0.059 ($6\varepsilon_y$)	$\gg$	0.038
LER	$2J_x$ [μm]	1.48	$\gg$	1.15
	$2J_y$ [μm]	0.036 ($6\varepsilon_y$)	<	0.044

Quest for source of emittance blowup

We had studied the following items:

- transverse wake field by accelerating tubes,
- higher order dispersion in BT,
- radiation excitation in BT,
- leakage of dispersion to straight section and energy jitter,
- jitter emittance,
- dispersion leakage into ECS accelerating tubes,
- coherent synchrotron radiation (CSR) effect by local microbunching,
- asymmetric dipole core in positron lattice was replaced with symmetric core.

Despite of tough efforts, the understanding and solution of BT emittance blowup issue were only partially achieved, so far.



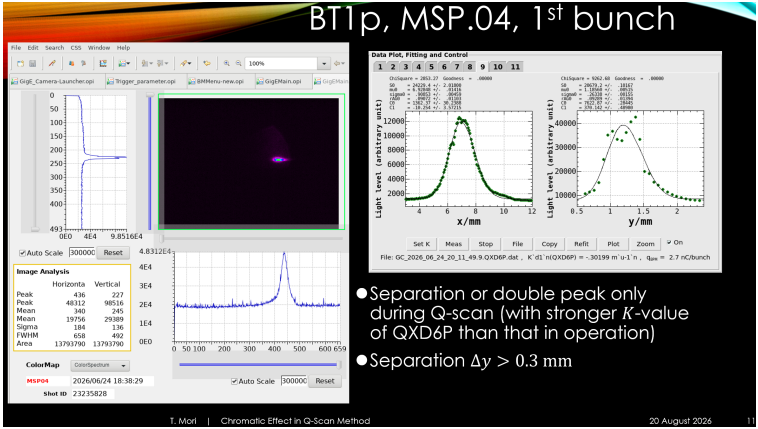
Recent highlighted items

- Hardware healthiness
  - Two quadrupoles where the unusual beam responses were observed
- Reconsideration of emittance blowup mechanism and data analysis
  - Orbit and energy jitter and the correlation of them

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Abnormal responses with Q-magnets



- Separation or double peak only during Q-scan (with stronger K-value of QXD6P than that in operation)
- Separation $\Delta y > 0.3$ mm

QXD6P

- Dispersion leakage?
- Hardware malfunction?

QAD4E is also picked.

- Abnormal response was observed with bump orbit test in a few years ago.

Magnet healthiness check

Magnet healthiness check were done by T. Ueda.

- Two magnets were selected because of the abnormal responses seen in the past experience.
- Measured items:
  - Common mode noise in a rack
  - Current-voltage linearity check
  - Layer short survey at operation current

Measured magnets

- QXD6P
- QAD4E



QAD4E(bottom)

磁極長	0.5 m
ボア半径	26㎜
鉄心重量	390kg
コイル重量	225kg
架台重量	100kg

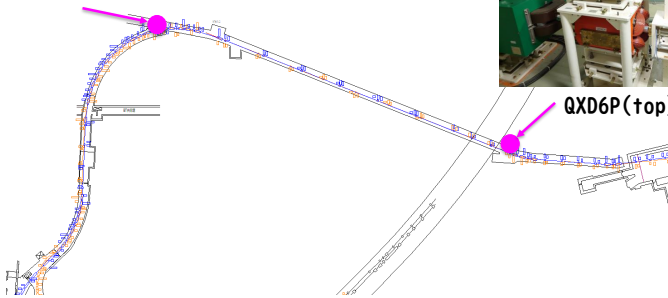
コイル	F-DOC 5.5㎜×4.5㎜
コイル巻数	90ターン/磁極
最大電流	30A
最大磁場勾配	16T/m
冷却抵抗	0.24Ω
全インダクタンス	0.5H

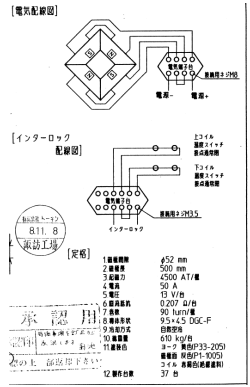
Power supply: 40V50A



QXD6P(top)

Power supply: 20V30A



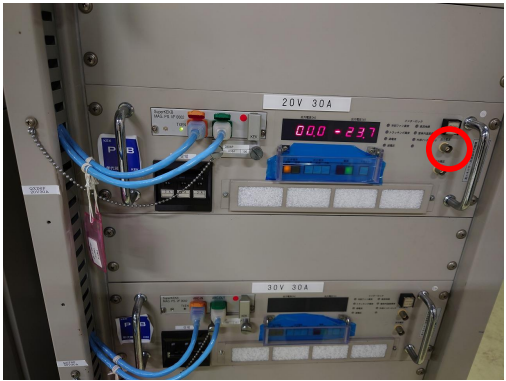


Common mode noise check on rack containing QXD6P

IDX　20V 30A

I_{mon} output at the front panel was used as voltage $(10\ \mathrm{V}/30\ \mathrm{A})$

Oscilloscope: RTA4004





Power supplies in Rack QS-2

QXF4P(20V30A)

QXD5P(20V30A)

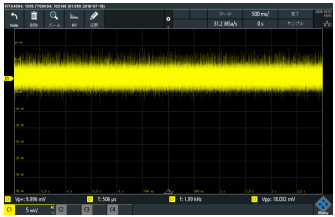
QXF5P(30V50A)

QXD6P(20V30A)

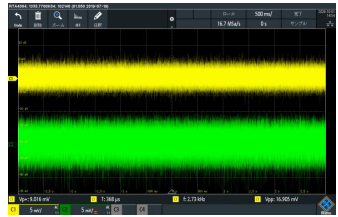
QXF6P(30V30A)

QXD7P(30V30A)

QXD6P(15.6A) pulse shape (single)

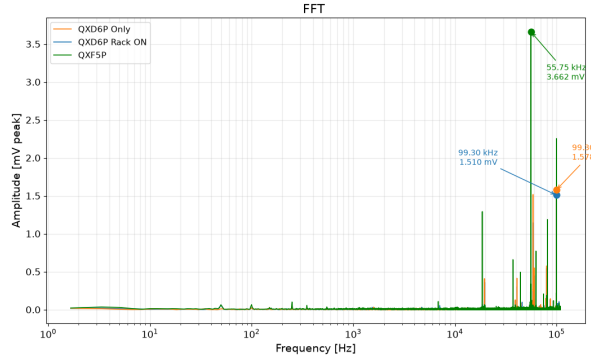


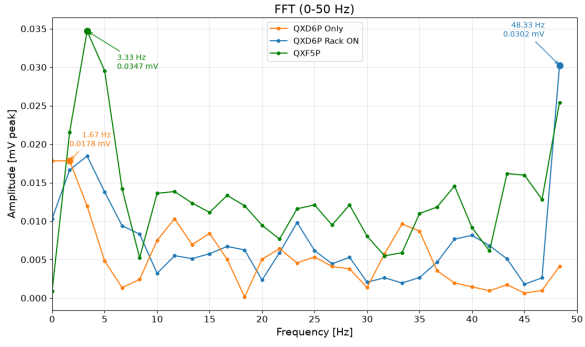
QXD6P (15.6A) with all others in QS-2 at max current



Green: QXF5P Imax

Peak is seen at 50 Hz but small enough.

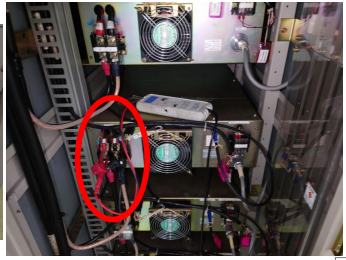




QXD6P current-voltage linearity check

PS: IDX 20V 30A

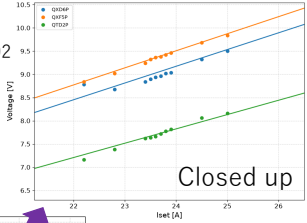


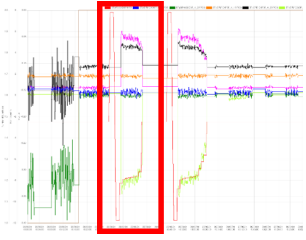


Output from back panel was measured.

Oscilloscope: RTA4004

Differential probe: RT-ZD002

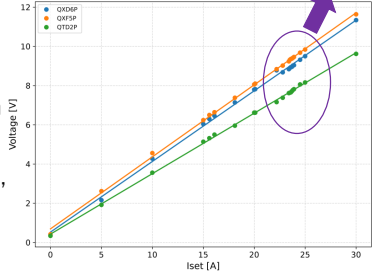




Operation current: 15.6A

Test currents were chosen from Q-scan period:

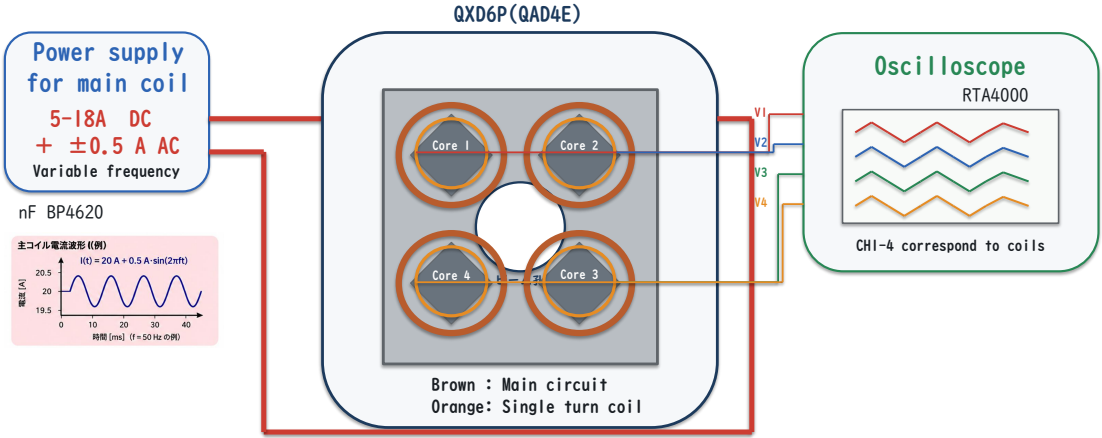
24.5, 23.9, 23.8, 23.7, 23.6, 23.5, 23.4, 22.8 A



Good linearity

- Slightly large fluctuation at 22.2 A in QXD6P
- Wasn't affected by other PS status in QS-2

Layer short survey for Q-magnets



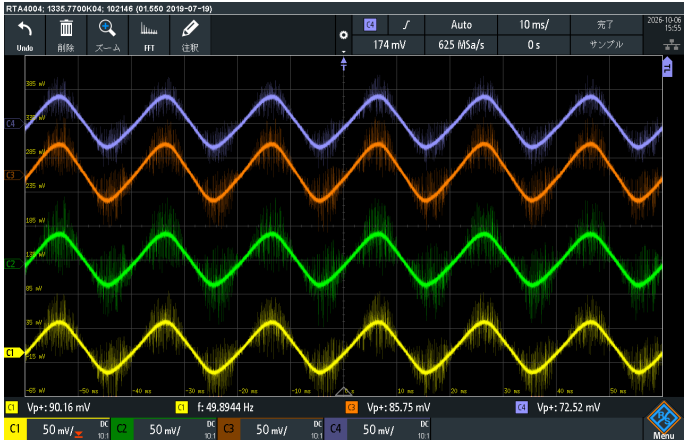
Measurement

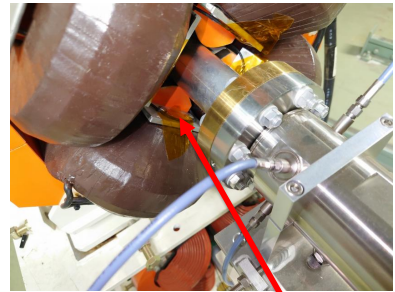
Main coil excitation : $I(t) = I_0 + 0.5\sin(2\pi ft)$ → ±0.5A, (50 or 1k)Hz oscillation applied in constant current
Voltages at single turn coils were measured

- Existence of layer short makes magnetic flux $\mathcal{O}(10\%)$ smaller which is detected at corresponding oscilloscope channel.
- Measurement system check at 50 Hz
- Layer short survey at 1 kHz

- QXD6P電磁石(QNPタイプ)

DC10A + AC ±0.5A 50Hz

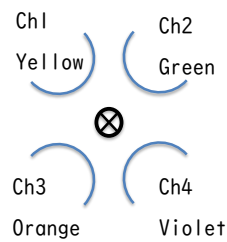




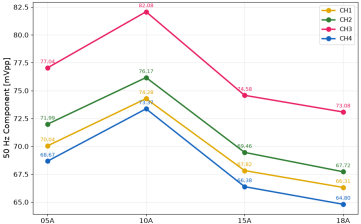
Detection coil

(single turn)

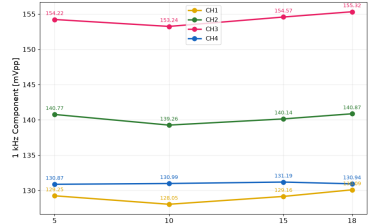
IE 1.25sq



AC ±0.5A **50Hz**



AC ±0.5A **1kHz**

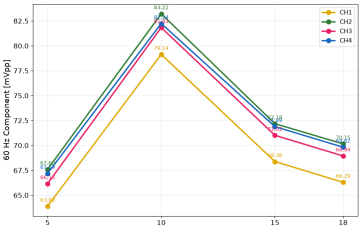


DC (5A, 10A, 15A, 18A)

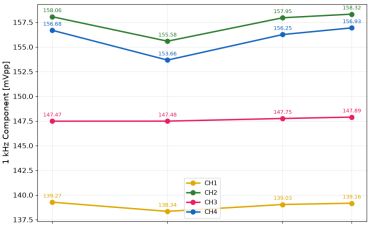
- QXD6P

Operation current: 15.6A

AC ±0.5A **60Hz**



AC ±0.5A **1kHz**



- QAD4E

Operation current: 26.7A

No layer short signal detected

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Introduction of quantities for BT emittance blowup study

- From the analogy of the betatron formalism, the extended state vector D and the jitter section covariance matrix Σ_D can be defined as:

$$D = \begin{pmatrix} \Delta & \Delta' & \delta_c \end{pmatrix}^\top \quad , \qquad \Sigma_D = \langle D D^\top \rangle \, . \tag{1}$$

- Therefore the total covariance is expressed as $\Sigma = \Sigma_\beta \oplus \Sigma_\eta \oplus \Sigma_D$,
- and the projected covariance can be written as $\Sigma_{\mathrm{proj}} = \Sigma_\beta + P_\eta \sigma_{\delta_\mathrm{b}} P_\eta^\top + P_{3\to2} \Sigma_D P_{3\to2}^\top$,
  with $P_\eta = \begin{pmatrix} \eta_{\mathrm{lat}} \\ \eta'_{\mathrm{lat}} \end{pmatrix}$, $\qquad P_{3\to2} = \begin{pmatrix} 1 & 0 & \eta_{\mathrm{meas}} \\ 0 & 1 & \eta'_{\mathrm{meas}} \end{pmatrix}$.
- Dispersion contribution is governed by the curly-H function: $\varepsilon^2_{\mathrm{proj}} = \varepsilon^2_\beta + \varepsilon_\beta \mathcal{H} \sigma^2_\delta$.
- The extended emittance $\varepsilon_D = \sqrt{\det(\Sigma_D)}$ which is independent of the dispersion was introduced for the beam diagnostics.
- Not only the jitter itself, the correlations between orbit and energy jitters remain a possible source of additional emittance growth.

Potential bias in dispersion correction

Standard assumption

The measured orbit response to an energy change is interpreted as the lattice dispersion:

$$\eta_{\mathrm{meas}}(s) = \frac{\partial \langle y(s) \rangle}{\partial \delta}.$$

Dispersion correction attempts to minimize $\eta_{\mathrm{meas}}(s)$.

Suppose the incoming beam coincidentally possesses a shot-to-shot covariance

$$\langle \Delta\, \delta_{\mathrm{c}} \rangle \neq 0, \qquad \langle \Delta'\, \delta_{\mathrm{c}} \rangle \neq 0,$$

at any locations in the beam line.

Possible issue

Then the measured orbit centroid may contain contributions from both

- lattice dispersion,
- beam-induced energy-correlated orbit jitter.

In this case,

$$\eta_{\mathrm{meas}} = \eta_{\mathrm{lat}} + \eta_{\mathrm{eff}},$$

where

$$\eta_{\mathrm{eff}} = \langle \Delta \delta_{\mathrm{c}} \rangle\, \sigma_{\delta_{\mathrm{c}}}^{-2},$$

η_{lat} and η_{eff} are not distinguishable from each other.

Possible consequence for dispersion correction

Dispersion correction is performed for a specific beam state:

$$\eta_{\mathrm{meas}} = \eta_{\mathrm{lat}} + \eta_{\mathrm{eff}}.$$

The correction minimizes η_{meas}.

Possible consequence for dispersion correction

Dispersion correction is performed for a specific beam state:

$$\eta_{\mathrm{meas}} = \eta_{\mathrm{lat}} + \eta_{\mathrm{eff}}.$$

The correction minimizes η_{meas}. This does not necessarily imply $\eta_{\mathrm{lat}} = 0$. Instead,

$$\eta_{\mathrm{lat}} = -\eta_{\mathrm{eff}}$$

satisfies the correction condition. The corrected optics should have the internal parameter η_{lat} which changes the linear projection P.
This effect is convolved as

$$\eta_{\mathrm{lat}}(s) = -\sum_n R_\eta(s, s_n)\eta_{\mathrm{eff}}(s_n). \tag{2}$$

Possible implications for Q-scan measurements

Remind the curly H function $\mathcal{H}$ in the projected emittance $\varepsilon_{\rm proj}$,

$$\mathcal{H} = \gamma\eta_{\rm lat}^2 + 2\alpha\eta_{\rm lat}\eta'_{\rm lat} + \beta{\eta'_{\rm lat}}^2, \tag{3}$$

where $(\eta_{\rm lat}, \eta'_{\rm lat})$ may cause possibly large $\mathcal{H}$.

Possible implications for Q-scan measurements

Remind the curly H function $\mathcal{H}$ in the projected emittance $\varepsilon_{\mathrm{proj}}$,

$$\mathcal{H} = \gamma\eta_{\mathrm{lat}}^2 + 2\alpha\eta_{\mathrm{lat}}\eta'_{\mathrm{lat}} + \beta{\eta'_{\mathrm{lat}}}^2, \tag{3}$$

where $(\eta_{\mathrm{lat}}, \eta'_{\mathrm{lat}})$ may cause possibly large $\mathcal{H}$. Possible consequences include

- increased contribution of $P\Sigma_D P^\top$,
- increased chromatic contribution through the curly-H term $\left(\varepsilon_\beta \mathcal{H}\sigma_{\delta_{\mathrm{b}}}^2\right)$,
- changes in the measured projected emittance.

Working hypothesis

Working hypothesis

Dispersion correction may depend weakly on the beam state through shot-to-shot centroid–energy correlations. If the centroid covariance matrix Σ_D changes, a residual lattice dispersion may become visible, leading to different projected-emittance measurements.

Further verification is required. Especially the key parameters are the correlation coefficients

$$\rho_{\Delta\delta_c}, \qquad \rho_{\Delta'\delta_c}. \tag{4}$$

Because

$$\langle \Delta\delta_c \rangle = \rho_{\Delta\delta_c}\sigma_\Delta\sigma_{\delta_c} \qquad \langle \Delta'\delta_c \rangle = \rho_{\Delta'\delta_c}\sigma_{\Delta'}\sigma_{\delta_c} \tag{5}$$

$$\eta_{\mathrm{eff}} = \rho_{\Delta\delta_c}\sigma_\Delta\sigma_{\delta_c}^{-1} \qquad \eta'_{\mathrm{eff}} = \rho_{\Delta'\delta_c}\sigma_{\Delta'}\sigma_{\delta_c}^{-1}. \tag{6}$$

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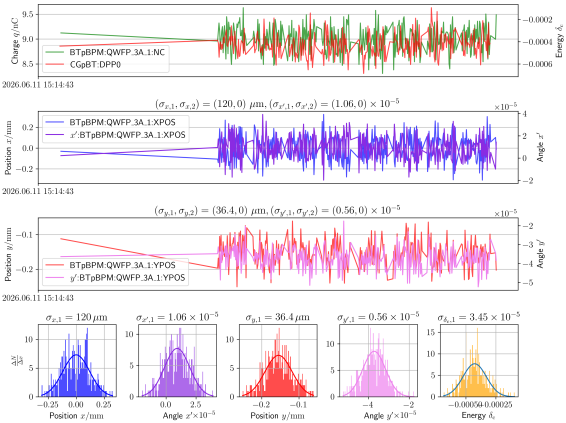
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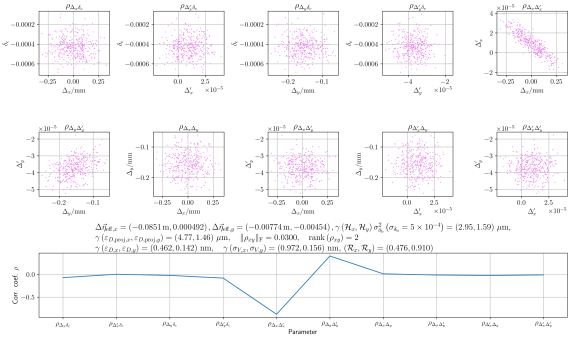
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Measured correlations

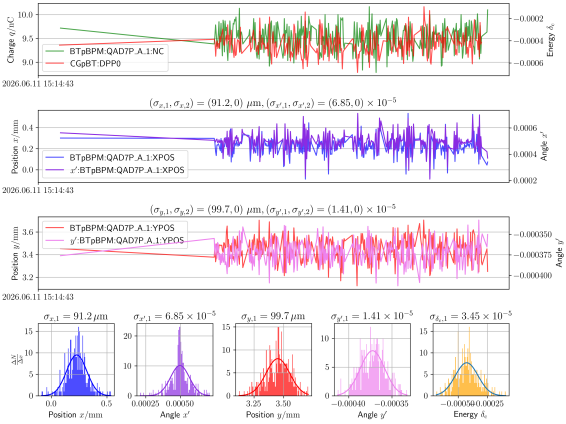
2026.06.11, Wire section

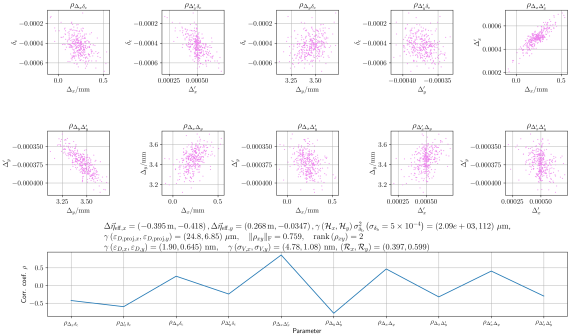




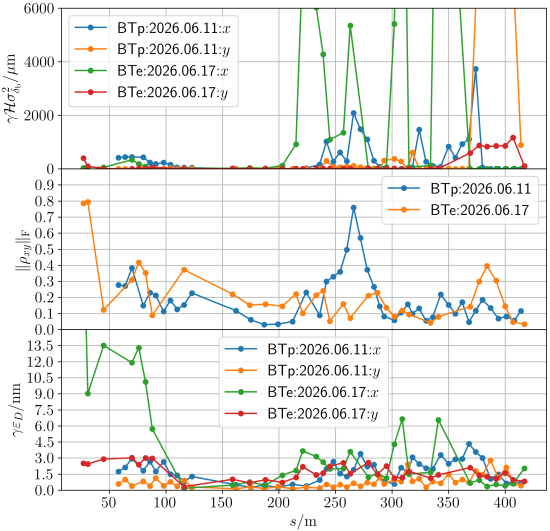
Measured correlations

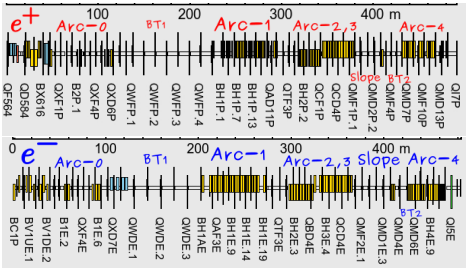
2026.06.11, in Arc-1

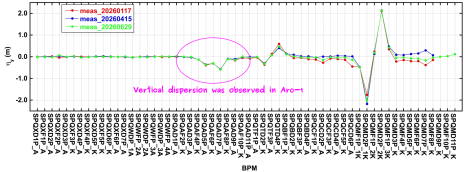




Result



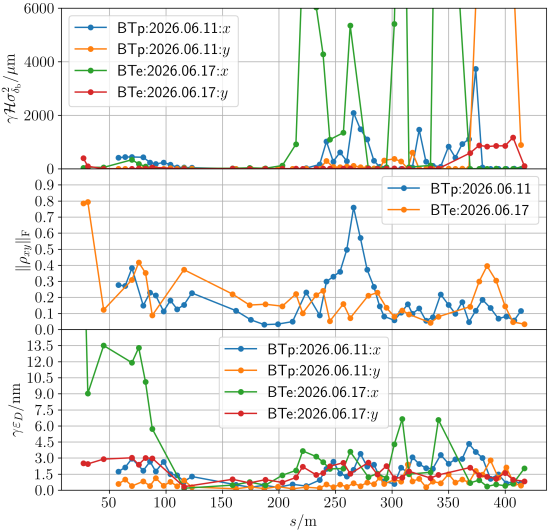




$$\|\rho_{xy}\|_{\mathrm{F}} = \sqrt{\rho^2_{\Delta_x\Delta_y} + \rho^2_{\Delta_x\Delta'_y} + \rho^2_{\Delta'_x\Delta_y} + \rho^2_{\Delta'_x\Delta'_y}}$$

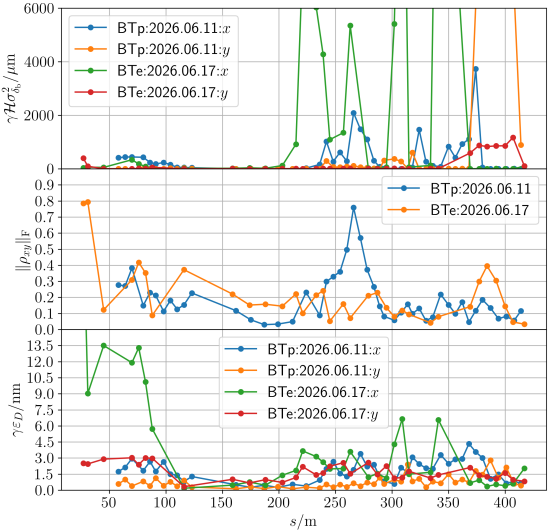
(Frobenius norm)

Discussion



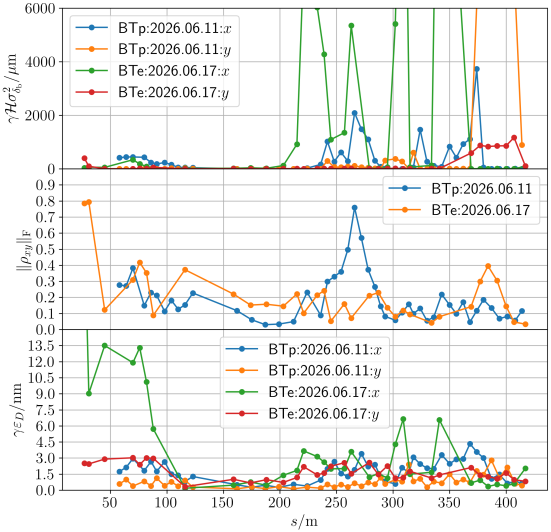
- Very stable $\mathcal{H}, \varepsilon_D$ and small x-y coupling in the straight section.

Discussion



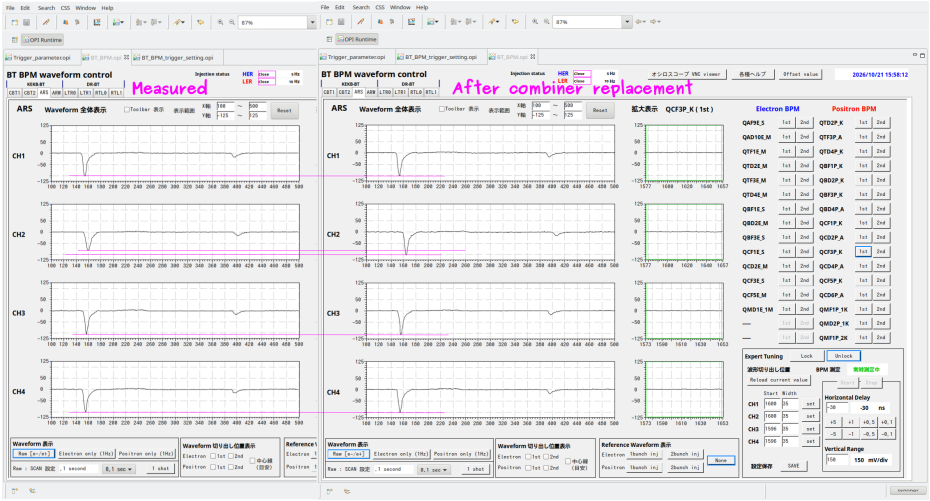
- Very stable $\mathcal{H}, \varepsilon_D$ and small x-y coupling in the straight section.
- Enhance of x-y coupling in arc sections
  - caused by skew quad?
  - Rather caused by existence of horizontal and vertical dispersions at the same location.

Discussion



- Very stable $\mathcal{H}, \varepsilon_D$ and small x-y coupling in the straight section.
- Enhance of x-y coupling in arc sections
  - caused by skew quad?
  - Rather caused by existence of horizontal and vertical dispersions at the same location.
- Structure of ε_D indicates existence of something out of linear optics.
  - Higher order optics?
    - In general, "higher order" of quantity becomes small, effect is small especially in single path lattices.
  - Lattice or BPM malfunction?

BT-BPM transmission line investigation



F. Ito, *et al.*

- Very recent result.
- Electrode orientation is like ⊗.
- Identical channels of 4-BPM electrodes are connected to a combiner.

- What if BPM response imbalance exists during dispersion correction?

BPM response imbalance scenario

- First order BPM response imbalance is expressed as $\begin{pmatrix}\Delta_x\\ \Delta_y\end{pmatrix}_{\mathrm{meas}} = \begin{pmatrix}1 & \kappa_y\\ \kappa_x & 1\end{pmatrix}\begin{pmatrix}\Delta_x\\ \Delta_y\end{pmatrix}_{\mathrm{true}}$, where $\kappa_{x,y}$ denote contamination factors.
- **Thus the fake dispersion will contaminate the dispersion measurement with** $\begin{pmatrix}\kappa_y\eta_y & \kappa_x\eta_x\end{pmatrix}^{\top}\delta_{\mathrm{c}}$.
- Invisible dispersion leakage will be introduced by dispersion correction, which cause larger $\mathcal{H}$.

The dispersion correction bias hypothesis is naturally realized with the BPM response imbalance.

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- BT emittance history
- Magnet healthiness check
- Reconsideration and consequences of emittance blowup mechanism
  - Introduction of quantities for BT emittance blowup study
  - Possible beam-state dependence of dispersion correction
- BPM data analysis: correlations and dispersions
  - Archived data analysis
  - BPM healthiness survey
  - BPM response imbalance scenario
- Summary
  - Summary and study proposal

Summary

- The magnet healthiness check was done. (T. Ueda)
  - No abnormal magnet was found.
- Extended state vector D and some quantities were introduced.
- The hypothesis on the potential bias in dispersion correction was proposed.
  - Important factor: $\rho_{\Delta\delta_c}, \rho_{\Delta'\delta_c}$.
  - But they are not easily measured.
  - The drift of η_{meas} would be the important indication.
- BPM jitter analysis was performed,
  - $\mathcal{H}$ is stable in straight section, not in dispersive section.
  - x-y coupling was seen,
  - ε_D is not conserved.
- BPM transmission line imbalance were found. (F. Ito, *et al.*)
- Dispersion correction bias hypothesis is naturally realized if the BPM induced x-y coupling exists.
  - Note that other candidates of η_{eff} or ε_D driver might also exist.

Study/development proposal

- Lattice malfunction
  - Raster scan at different phases.
  - Check the section with large ε_D.
- Dispersion correction bias hypothesis
  - Improvement of BPM accuracy and resolution (in progress).
  - Orbit jitter ($\sigma_\Delta, \sigma_{\Delta'}$) reduction.
    - Need to consider how it can be achieved.
  - Intentional dispersion leak test (in study list of next run).
  - ECS cavity voltage scan (in study list of next run).
  - Please let us know if there is any good idea.



Thank you for your attention!

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▷ Appendix A: Formalism for study on BT emittance blowup
- Basic consideration
- Introduction of extended emittance
- Summary of formalism

Motivation

- In BT-lines, the different measurements are used between upstream (BT1) and downstream (BT2) locations.
  - 4-profile measurement (Wire-scanner measurement in BT) at BT1
  - Quadrupole scan (Q-scan) at BT2
- Emittance measurements at BT1 and BT2 yield different projected emittances.
- The possibility of understanding emittance growth from the perspective of beam dynamics is explored.
  - What quantity is actually reconstructed?
  - How do dispersion and jitter contribute?
  - What is the correlation components and how do they contribute?
- Starting from linear dynamics, because the non-linear (higher order) effects are generally small for single path beamlines, except for sextupole fields, very large beta function, very strong convergence such as a final focus.

Basic linear formalism of beam propagation

- Transverse position:

$$\boldsymbol{x} = \boldsymbol{x}_\beta + \boldsymbol{x}_\eta = \begin{pmatrix} x_\beta + \eta_x \delta & x'_\beta + \eta'_x \delta \end{pmatrix}^\top, \tag{7}$$

  where the subscripts β, η denote that they are betatron and dispersion terms.
- Transportation from point 1 to 2: $\boldsymbol{x}_2 = R\boldsymbol{x}_1$
- Covariance matrices and projected emittance:

$$\Sigma_1 = \langle \boldsymbol{x}_1 \boldsymbol{x}_1^\top \rangle, \tag{8}$$

$$\Sigma_2 = R\Sigma_1 R^\top \tag{9}$$

$$\varepsilon^2_{\text{proj},i} = \det(\Sigma_i) = \langle x_i^2 \rangle \langle x_i'^2 \rangle - \langle x_i x_i' \rangle^2 = \varepsilon^2_{\beta,i} + \varepsilon_{\beta,i} \mathcal{H}_i \sigma^2_{\delta,i} \tag{10}$$

$$\mathcal{H} = \gamma\eta^2 + 2\alpha\eta\eta' + \beta\eta'^2 \qquad \text{(Curly H function)} \tag{11}$$

Another consistent expression

- There is another consistent expression with the transverse position and dispersion relation:

$$\boldsymbol{x} = \begin{pmatrix} x_\beta & x'_\beta & \delta \end{pmatrix}^\top \tag{12}$$

$$\Sigma = \langle \boldsymbol{x}\boldsymbol{x}^\top \rangle = \begin{pmatrix} \langle x_\beta^2 \rangle & \langle x_\beta x'_\beta \rangle & \langle x_\beta \delta \rangle \\ \langle x_\beta x'_\beta \rangle & \langle {x'_\beta}^2 \rangle & \langle x'_\beta \delta \rangle \\ \langle x_\beta \delta \rangle & \langle x'_\beta \delta \rangle & \langle \delta^2 \rangle \end{pmatrix} \tag{13}$$

  where the dispersion is naturally included through the x-δ correlation.

- Two expressions are connected with the matrix $P_{3\to 2}$ as follows:

$$P_{3\to 2} = \begin{pmatrix} 1 & 0 & \eta \\ 0 & 1 & \eta' \end{pmatrix} \tag{14}$$

$$\Sigma_{2\times 2} = \langle\!\langle (P_{3\to 2}\boldsymbol{x})(P_{3\to 2}\boldsymbol{x})^\top \rangle = \langle P_{3\to 2}\boldsymbol{x}\boldsymbol{x}^\top P_{3\to 2}^\top \rangle = P_{3\to 2}\Sigma_{3\times 3}P_{3\to 2}^\top \tag{15}$$

Introduction of centroid-jitter covariance

- Beam parameters which are not considered so far: jitters of the centroids of bunches
- Transverse (horizontal or vertical) orbit jitter can be defined as:

$$\boldsymbol{\Delta} = \begin{pmatrix} \Delta & \Delta' \end{pmatrix}^{\top}. \tag{16}$$

- Including energy jitter,

$$D = \begin{pmatrix} \Delta & \Delta' & \delta_{\mathrm{c}} \end{pmatrix}^{\top}. \tag{17}$$

- If the longitudinal position are also considered for the cavities, the jitter can be defined as

$$D = \begin{pmatrix} \Delta & \Delta' & z_{\mathrm{c}} & \delta_{\mathrm{c}} \end{pmatrix}^{\top}. \tag{18}$$

In the subsequent discussion of this report, Equation (17) is assumed.

Extended beam state vector

We introduce the extended state vector

$$\boldsymbol{x} = \begin{pmatrix} x_\beta & x'_\beta & \delta_\mathrm{b} & \Delta & \Delta' & \delta_\mathrm{c} \end{pmatrix}^\top, \tag{19}$$

where

- (x_β, x'_β) : betatron coordinates,
- δ_b : intrinsic energy spread,
- (Δ, Δ') : pulse-to-pulse orbit jitter,
- δ_c : pulse-to-pulse energy jitter.

The covariance matrix is

$$\Sigma_\mathrm{proj} = \left\langle \boldsymbol{x}_\mathrm{proj} \boldsymbol{x}_\mathrm{proj}^\top \right\rangle = \left\langle P_{6\to2} \boldsymbol{x}\boldsymbol{x}^\top P_{6\to2}^\top \right\rangle = P_{6\to2} \Sigma_{6\times6} P_{6\to2}^\top \tag{20}$$

Naive 6D covariance formulation

$$\left\langle \boldsymbol{x}\boldsymbol{x}^{\top} \right\rangle = \Sigma_{6\times 6} = \Sigma_{\beta} \oplus \Sigma_{\eta} \oplus \Sigma_{D} + \Sigma_{6\times 6,\mathrm{corr}} \tag{21}$$

$$= \begin{pmatrix} \Sigma_{\beta} & \Sigma_{\beta\eta} & \Sigma_{\beta D} \\ \Sigma_{\beta\eta} & \Sigma_{\eta} & \Sigma_{\eta D} \\ \Sigma_{\beta D} & \Sigma_{\eta D} & \Sigma_{D} \end{pmatrix} \tag{22}$$

$$= \begin{pmatrix} \left\langle x_{\beta}^{2} \right\rangle & \left\langle x_{\beta} x_{\beta}' \right\rangle & \left\langle x_{\beta}\delta_{\mathrm{b}} \right\rangle & \left\langle x_{\beta}\Delta_{x} \right\rangle & \left\langle x_{\beta}\Delta_{x}' \right\rangle & \left\langle x_{\beta}\delta_{\mathrm{c}} \right\rangle \\ \left\langle x_{\beta} x_{\beta}' \right\rangle & \left\langle x_{\beta}'^{2} \right\rangle & \left\langle x_{\beta}'\delta_{\mathrm{b}} \right\rangle & \left\langle x_{\beta}'\Delta_{x} \right\rangle & \left\langle x_{\beta}'\Delta_{x}' \right\rangle & \left\langle x_{\beta}'\delta_{\mathrm{c}} \right\rangle \\ \left\langle x_{\beta}\delta_{\mathrm{b}} \right\rangle & \left\langle x_{\beta}'\delta_{\mathrm{b}} \right\rangle & \left\langle \delta_{\mathrm{b}}^{2} \right\rangle & \left\langle \delta_{\mathrm{b}}\Delta_{x} \right\rangle & \left\langle \delta_{\mathrm{b}}\Delta_{x}' \right\rangle & \left\langle \delta_{\mathrm{b}}\delta_{\mathrm{c}} \right\rangle \\ \left\langle x_{\beta}\Delta_{x} \right\rangle & \left\langle x_{\beta}'\Delta_{x} \right\rangle & \left\langle \delta_{\mathrm{b}}\Delta_{x} \right\rangle & \left\langle \Delta_{x}^{2} \right\rangle & \left\langle \Delta_{x}\Delta_{x}' \right\rangle & \left\langle \Delta_{x}\delta_{\mathrm{c}} \right\rangle \\ \left\langle x_{\beta}\Delta_{x}' \right\rangle & \left\langle x_{\beta}'\Delta_{x}' \right\rangle & \left\langle \delta_{\mathrm{b}}\Delta_{x}' \right\rangle & \left\langle \Delta_{x}\Delta_{x}' \right\rangle & \left\langle \Delta_{x}'^{2} \right\rangle & \left\langle \Delta_{x}'\delta_{\mathrm{c}} \right\rangle \\ \left\langle x_{\beta}\delta_{\mathrm{c}} \right\rangle & \left\langle x_{\beta}'\delta_{\mathrm{c}} \right\rangle & \left\langle \delta_{\mathrm{b}}\delta_{\mathrm{c}} \right\rangle & \left\langle \Delta_{x}\delta_{\mathrm{c}} \right\rangle & \left\langle \Delta_{x}'\delta_{\mathrm{c}} \right\rangle & \left\langle \delta_{\mathrm{c}}^{2} \right\rangle \end{pmatrix}, \tag{23}$$

From the 6D covariance matrix to the projected beam matrix

- Two-level ensemble interpretation of $\Sigma_{6\times6}$: particle ensemble and bunch ensemble.
- Considering the particle ensemble in a single bunch, the jitter D is a constant vector.
- Therefore, $\langle x_\beta \delta_{\rm c} \rangle = \delta_{\rm c} \langle x_\beta \rangle = 0$ and same for all inter-level correlations.

Consequently, the correlation term disappears $\Sigma_{6\times6,\rm corr} = 0$, the covariance takes the block form

$$\Sigma = \Sigma_\beta \oplus \Sigma_\eta \oplus \Sigma_D = \begin{pmatrix} \Sigma_\beta & 0 & 0 \\ 0 & \sigma_{\delta_b}^2 & 0 \\ 0 & 0 & \Sigma_D \end{pmatrix},$$

with

$$\Sigma_D = \begin{pmatrix} \langle \Delta^2 \rangle & \langle \Delta\Delta' \rangle & \langle \Delta\delta_{\rm c} \rangle \\ \langle \Delta\Delta' \rangle & \langle \Delta'^2 \rangle & \langle \Delta'\delta_{\rm c} \rangle \\ \langle \Delta\delta_{\rm c} \rangle & \langle \Delta'\delta_{\rm c} \rangle & \langle \delta_{\rm c}^2 \rangle \end{pmatrix}.$$

Projection into transverse phase space

The measured transverse coordinates are

$$\begin{pmatrix} x \\ x' \end{pmatrix} = P_{6\to 2}\boldsymbol{x},$$

with

$$P_{6\to 2} = \begin{pmatrix} 1 & 0 & \eta & 1 & 0 & \eta \\ 0 & 1 & \eta' & 0 & 1 & \eta' \end{pmatrix}. \tag{24}$$

The projected covariance matrix becomes

$$\Sigma_{\mathrm{proj}} = P_{6\to 2}\Sigma P_{6\to 2}^{\top}.$$

After expansion,

$$\boxed{\Sigma_{\mathrm{proj}} = \Sigma_{\beta} + \sigma_{\delta_b}^2 \boldsymbol{\eta}\boldsymbol{\eta}^{\top} + P_{3\to 2}\Sigma_D P_{3\to 2}^{\top}} \tag{25}$$

where

$$\boldsymbol{\eta} = \begin{pmatrix} \eta \\ \eta' \end{pmatrix}, \quad P_{3\to 2} = \begin{pmatrix} 1 & 0 & \eta \\ 0 & 1 & \eta' \end{pmatrix}.$$

Jitter vector of mass center

- Also the effect from the mass centroid jitter can be calculated as:

$$D \equiv \begin{pmatrix} \Delta & \Delta' & \delta \end{pmatrix}^{\top} \tag{26}$$

$$\Sigma_D = \langle D D^{\top} \rangle \tag{27}$$

- Additional beam size squared term is expressed as follows:

$$\sigma_D^2(K) = (r_D(K))^{\top} \Sigma_D r_D(K)\,, \tag{28}$$

where $r_D(K)$ is denoted with using transfer matrix R as:

$$r_D(K) = \begin{pmatrix} R_{k\,k}(K) & R_{k\,k+1}(K) & R_{k\,k}(K)\,\eta + R_{k\,k+1}(K)\,\eta' \end{pmatrix}^{\top} \qquad (k = 1, 3) \tag{29}$$

Introduction of extended emittance

- For checking stability of the beam condition along the beamline and time,
- an effective scalar quantity 'extended emittance' denoted as ε_D is introduced.

$$\begin{aligned}\det(\Sigma_D) &= \langle\Delta^2\rangle\langle\Delta'^2\rangle\langle\delta_c^2\rangle + 2\langle\Delta\Delta'\rangle\langle\Delta\delta_c\rangle\langle\Delta'\delta_c\rangle \\ &\quad - \langle\Delta^2\rangle\langle\Delta'\delta_c\rangle^2 - \langle\Delta'^2\rangle\langle\Delta\delta_c\rangle^2 - \langle\delta_c^2\rangle\langle\Delta\Delta'\rangle^2 \qquad (30)\\ &= \sigma_\Delta^2\sigma_{\Delta'}^2\sigma_{\delta_c}^2\left(1 + 2\rho_{\Delta\Delta'}\rho_{\Delta\delta_c}\rho_{\Delta'\delta_c} - \rho_{\Delta\delta_c}^2 - \rho_{\Delta'\delta_c}^2 - \rho_{\Delta\Delta'}^2\right) \qquad (31)\\ \varepsilon_D &= \sqrt{\det(\Sigma_D)} = \sigma_\Delta\sigma_{\Delta'}\sigma_{\delta_c}\left(1 + 2\rho_{\Delta\Delta'}\rho_{\Delta\delta_c}\rho_{\Delta'\delta_c} - \rho_{\Delta\delta_c}^2 - \rho_{\Delta'\delta_c}^2 - \rho_{\Delta\Delta'}^2\right)^{\frac{1}{2}} \qquad (32)\end{aligned}$$

- Σ_D propagates like an usual beam covariance matrix, such as $\Sigma_{D,2} = R\Sigma_{D,1}R^\top$, where R denotes the transfer matrix.
- If R is a symplectic, ε_D is conserved all along the beamline except sections including
  - ECS or RF cavity,
  - jitter-intrinsic energy spread correlation,
  - non-linear effects.

Important property of extended emittance

The true jitter vector D is not the measured jitter vector D_{meas} because of the existence of dispersion η_{meas}.

$$D_{\mathrm{meas}} = (\Delta + \eta_{\mathrm{meas}}\delta_{\mathrm{c}}, \Delta' + \eta_{\mathrm{meas}}\delta_{\mathrm{c}})^{\top}. \quad (33)$$

The dispersion contamination operator C_η can be defined and D_{meas} is expressed as

$$C_\eta = \begin{pmatrix} 1 & 0 & \eta_{\mathrm{meas}} \\ 0 & 1 & \eta'_{\mathrm{meas}} \\ 0 & 0 & 1 \end{pmatrix} \quad (34)$$

$$D_{\mathrm{meas}} = C_\eta D \quad (35)$$

Thus the following formalism is true:

$$\det(C_\eta) = 1 \quad (36)$$

$$\Sigma_D^{\mathrm{meas}} = C_\eta \Sigma_D^{\mathrm{meas}} C_\eta^{\top} \quad (37)$$

$$\varepsilon_D^2 = \det(\Sigma_D^{\mathrm{meas}}) = \det(\Sigma_D) \quad (38)$$

Consequently, ε_D is insensitive to $\eta_{\mathrm{meas}}, \eta_{\mathrm{lat}}, \eta_{\mathrm{eff}}$ within the frame of linear optics. It would be the nice diagnostics to check the ε_D constancy along the beamline and time.

Important parameters: $\sigma_V = \sigma_\Delta \sigma_{\Delta'} \sigma_{\delta_c}, \mathcal{R} = \dfrac{\sqrt{\det(\Sigma_D)}}{\sigma_V}$. Thanks to the small σ_{δ_c}, even if the measured values are substituted in σ_V and $\mathcal{R}$, the quantities are expected to be modulated only about a few percent at maximum. Strategy to check the hypothesis is as follows:

- Reduce σ_V, increase $\mathcal{R}$,
- perform dispersion correction,
- perform emittance measurement.

What was learnt

- From the analogy of the betatron formalism, the extended state vector D and the jitter section covariance matrix Σ_D can be defined as:

$$D = \begin{pmatrix} \Delta & \Delta' & \delta_c \end{pmatrix}^\top \quad , \qquad \Sigma_D = \langle DD^\top \rangle \, . \tag{39}$$

- Therefore the total covariance is expressed as

$$\Sigma = \Sigma_\beta \oplus \Sigma_\eta \oplus \Sigma_D, \tag{40}$$

- and the projected covariance can be written as

$$\Sigma_{\rm proj} = \Sigma_\beta + P_\eta \sigma_{\delta_{\rm b}} P_\eta^\top + P_{3\to 2} \Sigma_D P_{3\to 2}^\top. \tag{41}$$

- Dispersion contribution is governed by the curly-H function: $\varepsilon_{\rm proj}^2 = \varepsilon_\beta^2 + \varepsilon_\beta \mathcal{H} \sigma_\delta^2$.
- The extended emittance $\varepsilon_D = \sqrt{\det(\Sigma_D)}$ which is independent of the dispersion was introduced for the beam diagnostics.
- Not only the jitter itself, the correlations between orbit and energy jitters remain a possible source of additional emittance growth.